21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY ROUND I ALTERNATE TOPIC PACKET 08 FOURIER ANALYSIS, POINCARÉ INEQUALITY, AND INFLUENCES ON THE BOOLEAN CUBE

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ABSTRACT. **Organizing question:** How does Fourier analysis convert coordinate sensitivity into a variance bound? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

At a glance

Status	Alternate Round I topic
Setting	Discrete product space $\{-1,1\}^n$
Primary tools	Walsh–Fourier basis, Parseval, discrete derivatives, influences
Background	Linear algebra; finite probability; elementary combinatorics
Relative difficulty	3/4
Principal perspective	Discrete probability and theoretical computer science

1. Why this topic may appeal to you

Choose this alternate if you are interested in discrete mathematics, theoretical computer science, learning theory, or Boolean functions. It gives a finite-dimensional Fourier theory parallel to the cycle calculation but emphasizes coordinate sensitivity rather than geometry in physical space.

2. Fixed setting

Let $\Omega = \{-1,1\}^n$ with uniform measure. For $S \subseteq [n]$, define

$$\chi_S(x) = \prod_{i\in S} x_i, \qquad D_i f(x) = \frac{f(x) - f(x^{\oplus i})}{2},$$

where $x^{\oplus i}$ flips coordinate i.

3. Principal theorem

Develop

$$f = \sum_{S\subseteq[n]} \widehat{f}(S)\chi_S$$

Date: Fall 2026.

and prove

$$\mathrm{Var}(f) = \sum_{S\neq\varnothing} \widehat{f}(S)^2, \qquad \mathbf{E}[(D_i f)^2] = \sum_{S\ni i} \widehat{f}(S)^2.$$

Conclude

$$\mathrm{Var}(f) \leq \sum_{i=1}^{n} \mathbf{E}[(D_i f)^2].$$

For $f : \Omega \to \{-1, 1\}$, show

$$\mathbf{E}[(D_i f)^2] = \mathbf{P}(f(X) \neq f(X^{\oplus i})) =: \mathrm{Inf}_i(f).$$

4. Required proof content

Prove orthonormality of Walsh characters, Fourier inversion, and Parseval. Compute $D_i\chi_S$ and derive the influence identity. Identify equality cases in the Poincare inequality.

5. Applications, exercises, and pitfalls

Compute the total influence of dictators and parity and analyze majority in small dimensions. Explain the conclusion for a balanced Boolean function. Exercises may ask for the Fourier expansion of a two-bit function or influence computations. Do not mix $\{0, 1\}$ and $\{-1, 1\}$ conventions or derivative normalizations.

6. What may be taken as input

You may use finite-dimensional orthogonal expansion once the basis and normalization have been established. Hypercontractivity is not part of this assignment.

7. Suggested reading route and attribution

(1) R. O'Donnell, *Analysis of Boolean Functions*, Chapter 1 for Fourier expansion and Chapter 2, especially Sections 2.2–2.4, for influences and basic inequalities. Public author copy; arXiv edition.

(2) R. van Handel, *Probability in High Dimension*, Section 2.1 for the product-space variance viewpoint (public course notes).

8. Before the required consultation

Bring a completed one-page preparation document containing: the exact theorem statements; a dependency map of the proof; the portion you will prove in full; planned examples; two exercises with solutions; exact source locations; anticipated questions; and any point you do not yet understand. The consultation is for resolving mathematical and expository issues, not for choosing a topic from scratch.

9. AI-use disclosure

The preparation document must disclose any use of generative artificial-intelligence tools. Identify the tool, the task for which it was used, material incorporated into the presentation, and how mathematical assertions and references were independently verified. You remain responsible for every statement in the talk.

This is an instructor-authored assignment guide, not a substitute for reading the listed sources. All normalizations in the talk should agree with this packet unless a change is explicitly approved in advance.