21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY ROUND I PRINCIPAL TOPIC PACKET 07 LOGARITHMIC SOBOLEV INEQUALITIES AND THE HERBST ARGUMENT

MATTHEW ROSENZWEIG

Abstract. **Organizing question:** How does a logarithmic Sobolev inequality produce Gaussian concentration? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

At a glance

Status	Principal Round I topic
Setting	Probability measures satisfying a logarithmic Sobolev inequality; Gaussian model
Primary tools	Functional entropy, exponential moments, differentiation, Chernoff optimization
Background	Calculus; probability; comfort manipulating expectations
Relative difficulty	4/4
Principal perspective	Information theory, probability, and concentration

1. Why this topic may appeal to you

This topic gives one of the central implication chains in modern probability: a local energy inequality controls exponential moments and hence rare fluctuations. It has the strongest information-theoretic flavor among the principal topics.

2. Fixed functional inequality

Assume a probability measure μ on $\mathbb{R}^d$ satisfies

$$\mathrm{Ent}_\mu(e^g) \le \frac{C}{2} \int e^g |\nabla g|^2 \, \mathrm{d}\mu$$

for smooth g, where

$$\mathrm{Ent}_\mu(h) = \int h \log h \, \mathrm{d}\mu - \left(\int h \, \mathrm{d}\mu \right) \log \left(\int h \, \mathrm{d}\mu \right).$$

For standard Gaussian measure, $C = 1$ in this normalization.

Date: Fall 2026.

3. Principal theorem

If F is L-Lipschitz, prove

$$\log \int e^{\lambda(F - \int F \, \mathrm{d}\mu)} \, \mathrm{d}\mu \le \frac{CL^2\lambda^2}{2}, \qquad \lambda \in \mathbb{R},$$

and hence

$$\mu\left(F - \int F \, \mathrm{d}\mu \ge r\right) \le \exp\left(-\frac{r^2}{2CL^2}\right).$$

Apply the same argument to $-F$ for the two-sided bound.

4. Required proof content

(1) Set $\psi(\lambda) = \log \int e^{\lambda F} \, \mathrm{d}\mu$ and prove

$$\lambda\psi'(\lambda) - \psi(\lambda) = \frac{\mathrm{Ent}_\mu(e^{\lambda F})}{\int e^{\lambda F} \, \mathrm{d}\mu}.$$

(2) Apply the logarithmic Sobolev inequality to $g = \lambda F$ and obtain a differential inequality for $\psi(\lambda)/\lambda$.
(3) Integrate from 0 to λ, using $\lim_{\lambda\to 0} \psi(\lambda)/\lambda = \int F \, \mathrm{d}\mu$.
(4) Apply exponential Markov inequality and optimize in λ.
(5) Explain the approximation needed for a Lipschitz but nonsmooth observable such as $F(x) = |x|$.

5. Required examples and interpretation

Apply the theorem to the Euclidean norm of a standard Gaussian vector and explain the thin-shell conclusion. Include another Lipschitz observable, such as a coordinate maximum or distance to a set. At theorem-map level, explain that logarithmic Sobolev inequalities also convert entropy production into exponential relative-entropy decay.

6. What may be taken as input

The Gaussian logarithmic Sobolev inequality may be taken as a theorem; its proof is not part of Round I. You may first work with smooth bounded F and then discuss truncation and approximation.

7. Exercises and common pitfalls

Exercises may ask for Chernoff optimization, the two-sided tail, or a shell bound. Do not confuse thermodynamic/Shannon entropy with the functional entropy $\mathrm{Ent}_\mu(h)$. Track C and L throughout. A Poincare inequality alone does not yield the same Gaussian tail by this argument.

8. Suggested reading route and attribution

(1) R. van Handel, *Probability in High Dimension*, Section 3.3, approximately pp. 55–61, for the entropy method and Herbst-type argument (public course notes).
(2) D. Chafaï and J. Lehec, *Logarithmic Sobolev Inequalities Essentials*, Section 1.3 and Section 3.1, approximately pp. 11–14 and 35–37 (public lecture notes).
(3) M. Ledoux, "Concentration of measure and logarithmic Sobolev inequalities," *Seminaire de Probabilites XXXIII*, Lecture Notes in Mathematics 1709 (1999), 120–216, especially the Laplace-transform/Herbst argument. DOI; public EuDML copy.

(4) D. Chafaï, *Mini-course on logarithmic Sobolev inequalities*, 2024, especially Section 4 (public mini-course notes).

9. Before the required consultation

Bring a completed one-page preparation document containing: the exact theorem statements; a dependency map of the proof; the portion you will prove in full; planned examples; two exercises with solutions; exact source locations; anticipated questions; and any point you do not yet understand. The consultation is for resolving mathematical and expository issues, not for choosing a topic from scratch.

10. AI-use disclosure

The preparation document must disclose any use of generative artificial-intelligence tools. Identify the tool, the task for which it was used, material incorporated into the presentation, and how mathematical assertions and references were independently verified. You remain responsible for every statement in the talk.

This is an instructor-authored assignment guide, not a substitute for reading the listed sources. All normalizations in the talk should agree with this packet unless a change is explicitly approved in advance.