21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY ROUND I PRINCIPAL TOPIC PACKET 06 GAUSSIAN POINCARÉ INEQUALITY FROM THE ORNSTEIN–UHLENBECK SEMIGROUP

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Abstract. **Organizing question:** Why is the Gaussian Poincaré inequality dimension-free? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

At a glance

Status	Principal Round I topic
Setting	Continuous probability: standard Gaussian measure on $\mathbb{R}^d$
Primary tools	Mehler formula, Gaussian integration by parts, semigroup interpolation, Jensen
Background	Multivariable calculus; Gaussian random vectors; basic semigroup language
Relative difficulty	3/4
Principal perspective	Probability, dynamics, and high-dimensional analysis

1. Why this topic may appeal to you

This is the first genuinely high-dimensional continuous model. It gives a complete semigroup proof, exhibits several reusable identities, and explains why independent Gaussian coordinates do not force deterioration of the Poincare constant.

2. Fixed normalization

Let γ_d be standard Gaussian measure on $\mathbb{R}^d$ and define

$$P_t f(x) = \mathbf{E}\left[f\left(e^{-t}x + \sqrt{1-e^{-2t}}\,G\right)\right], \qquad G \sim \gamma_d.$$

The generator is $L = \Delta - x \cdot \nabla$.

3. Principal theorem

For smooth f with sufficient growth control,

$$\mathrm{Var}_{\gamma_d}(f) \le \int_{\mathbb{R}^d} |\nabla f|^2 \, \mathrm{d}\gamma_d.$$

The constant 1 is sharp, independent of d, and attained by nonconstant affine functions.

Date: Fall 2026.

4. Required proof content

(1) Prove Gaussian invariance of the Mehler formula.
(2) Prove the commutation identity $\nabla P_t f = e^{-t} P_t(\nabla f)$.
(3) Establish Gaussian integration by parts and deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\int (P_t f)^2 \,\mathrm{d}\gamma_d = -2\int |\nabla P_t f|^2 \,\mathrm{d}\gamma_d.$$

(4) Show $P_t f \to \int f \,\mathrm{d}\gamma_d$ and derive

$$\mathrm{Var}_{\gamma_d}(f) = 2\int_0^\infty \int |\nabla P_t f|^2 \,\mathrm{d}\gamma_d \,\mathrm{d}t.$$

(5) Use Jensen and invariance to prove

$$\int |\nabla P_t f|^2 \,\mathrm{d}\gamma_d \le e^{-2t}\int |\nabla f|^2 \,\mathrm{d}\gamma_d,$$

then integrate.

5. Interpretation and comparison

Explain exponential contraction of nonconstant observables under Ornstein–Uhlenbeck evolution and the meaning of a dimension-free constant. Compare this proof conceptually with tensorization. Dimension-free does not mean every variance is bounded independently of d: the gradient energy may grow with d.

6. What may be taken as input

You may use the semigroup property after checking it from Gaussian covariance. Approximation beyond smooth bounded functions may be described rather than proved. The five identities above are the required proof.

7. Exercises and common pitfalls

Possible exercises include Gaussian integration by parts in one dimension, equality for affine functions, or exponential variance decay. Do not write $P_t(|\nabla f|)^2 \le P_t(|\nabla f|^2)$ without identifying Jensen's inequality. Keep the generator and Mehler normalizations consistent.

8. Suggested reading route and attribution

(1) D. Chafaï and J. Lehec, *Logarithmic Sobolev Inequalities Essentials*, Sections 1.2–1.4, approximately pp. 8–17, for the Ornstein–Uhlenbeck semigroup, Gaussian Poincare inequality, and convergence to equilibrium (public lecture notes).
(2) R. van Handel, *Probability in High Dimension*, Sections 2.2–2.4, approximately pp. 21–40 (public course notes).
(3) D. Bakry, I. Gentil, and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, pp. 177–233, as an instructor-level reference. DOI and publisher page.

9. Before the required consultation

Bring a completed one-page preparation document containing: the exact theorem statements; a dependency map of the proof; the portion you will prove in full; planned examples; two exercises with solutions; exact source locations; anticipated questions; and any point you do not yet understand. The consultation is for resolving mathematical and expository issues, not for choosing a topic from scratch.

10. AI-use disclosure

The preparation document must disclose any use of generative artificial-intelligence tools. Identify the tool, the task for which it was used, material incorporated into the presentation, and how mathematical assertions and references were independently verified. You remain responsible for every statement in the talk.

This is an instructor-authored assignment guide, not a substitute for reading the listed sources. All normalizations in the talk should agree with this packet unless a change is explicitly approved in advance.