

21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY
ROUND I PRINCIPAL TOPIC PACKET 05
THE NASH INEQUALITY AND HEAT-FLOW SMOOTHING

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ABSTRACT. Organizing question: How does a scale-sensitive inequality produce algebraic decay and smoothing? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

AT A GLANCE

Status	Principal Round I topic
Setting	Continuous/PDE: the heat equation on the real line
Primary tools	One-dimensional calculus, interpolation, energy estimates, scalar ODEs
Background	Single-variable analysis; elementary PDE notation
Relative difficulty	3/4
Principal perspective	Analysis/PDE and scale-sensitive functional inequalities

1. WHY THIS TOPIC MAY APPEAL TO YOU

This topic is the cleanest way to see the branch of the course that is not organized around an equilibrium probability measure. It explains why whole-space heat flow decays algebraically rather than exponentially and why scaling determines the exponents.

2. FIXED SETTING AND PRINCIPAL INEQUALITIES

Work first with $f \in C_c^\infty(\mathbb{R})$. Prove

$$\|f\|_\infty^2 \leq 2 \|f\|_2 \|f'\|_2$$

and deduce the one-dimensional Nash inequality

$$\|f\|_2^6 \leq 4 \|f\|_1^4 \|f'\|_2^2.$$

The constant is secondary; the exponents and scaling are essential.

3. REQUIRED HEAT-FLOW APPLICATION

For $u_t = u_{xx}$ on $\mathbb{R}$, set $y(t) = \|u(t)\|_2^2$ and $M = \|u(0)\|_1$. Using L^1 contraction and the energy identity, prove

$$y'(t) = -2 \|u_x(t)\|_2^2 \leq -\frac{y(t)^3}{2M^4}.$$

Solve the differential inequality and obtain

$$\|u(t)\|_2 \leq Mt^{-1/4}, \quad t > 0,$$

up to the harmless constant dictated by your normalization. State the resulting $L^1 \rightarrow L^2$ operator estimate.

4. REQUIRED PROOF CONTENT

- (1) Prove the L^∞ estimate from the fundamental theorem of calculus applied to f^2 .
- (2) Derive Nash using $\|f\|_2^2 \leq \|f\|_1 \|f\|_\infty$.
- (3) Check every scaling exponent under $f_a(x) = f(ax)$ and use the same scaling to explain the failure of a whole-line Poincare inequality with a fixed constant.
- (4) Derive the heat energy identity and solve the scalar differential inequality without skipping the algebra.
- (5) Compare algebraic Nash decay on $\mathbb{R}$ with exponential Poincare decay on a compact interval or circle.

5. WHAT MAY BE TAKEN AS INPUT

You may take L^1 contraction of the heat semigroup as an input, but explain it through the heat kernel and Young's inequality or through positivity and mass preservation. The explicit heat kernel is a check, not a replacement for the Nash proof.

6. OPTIONAL EXTENSION, EXERCISES, AND PITFALLS

Using self-adjointness and the semigroup property, derive an $L^1 \rightarrow L^\infty$ bound of order $t^{-1/2}$. Exercises may address the scalar ODE, dimension- d scaling, or this semigroup step. Distinguish decay of a norm from decay of its square; do not conceal the use of L^1 contraction; and do not call Nash a Poincare inequality.

7. SUGGESTED READING ROUTE AND ATTRIBUTION

- (1) T. H. Colding, notes prepared by D. Winter, *Introduction to Partial Differential Equations*, MIT OpenCourseWare 18.152, Lecture 13, "The Heat Equation," for undergraduate heat-equation background ([course page](#)).
- (2) J. Nash, "Continuity of solutions of parabolic and elliptic equations," *American Journal of Mathematics* 80 (1958), no. 4, 931–954. [DOI and journal record](#). This is the historical source; the talk uses a much simpler one-dimensional route.
- (3) D. Bakry, I. Gentil, and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, the chapter "Sobolev Inequalities," pp. 277–343, for the general relation among Sobolev/Nash inequalities, semigroup bounds, and heat kernels. [DOI and publisher page](#).

8. BEFORE THE REQUIRED CONSULTATION

Bring a completed one-page preparation document containing: the exact theorem statements; a dependency map of the proof; the portion you will prove in full; planned examples; two exercises with solutions; exact source locations; anticipated questions; and any point you do not yet understand. The consultation is for resolving mathematical and expository issues, not for choosing a topic from scratch.

9. AI-USE DISCLOSURE

The preparation document must disclose any use of generative artificial-intelligence tools. Identify the tool, the task for which it was used, material incorporated into the presentation, and how mathematical assertions and references were independently verified. You remain responsible for every statement in the talk.

This is an instructor-authored assignment guide, not a substitute for reading the listed sources. All normalizations in the talk should agree with this packet unless a change is explicitly approved in advance.