21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY
ROUND I PRINCIPAL TOPIC PACKET 04
CHEEGER INEQUALITY AND BOTTLENECKS ON REGULAR GRAPHS

MATTHEW ROSENZWEIG

ABSTRACT. **Organizing question:** How does a bottleneck force slow convergence to equilibrium? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

AT A GLANCE

Status	Principal Round I topic
Setting	Discrete geometry: finite connected regular graphs
Primary tools	Rayleigh quotients, level sets, discrete coarea, Cauchy–Schwarz
Background	Graph theory; finite Markov chains; comfort with multi-step proofs
Relative difficulty	4/4
Principal perspective	Discrete geometry, probability, and spectral theory

1. WHY THIS TOPIC MAY APPEAL TO YOU

This is the most geometric discrete topic and one of the most demanding Round I choices. It converts a global analytic quantity, the spectral gap, into a statement about cuts and bottlenecks. The proof contains a genuine mechanism rather than only an exact calculation.

2. FIXED NORMALIZATION

Let $G = (V, E)$ be a finite connected d-regular graph. Let P be the simple-random-walk transition matrix, $L = P - I$, and π uniform. Then

$$\mathcal{E}(f, f) = \frac{1}{d|V|} \sum_{\{x,y\}\in E} (f(x) - f(y))^2.$$

Define

$$\Phi := \min_{\varnothing\neq S\subseteq V,\ |S|\leq|V|/2} \frac{|\partial_E S|}{d|S|},$$

where $\partial_E S$ consists of edges with exactly one endpoint in S. Let λ be the spectral gap of $I - P$.

Date: Fall 2026.

3. Principal theorem

Prove

$$\frac{\Phi^2}{2} \le \lambda \le 2\Phi.$$

The upper estimate says a sparse cut yields a low-energy test function. The lower estimate says a low-energy function has a level set giving a sparse cut.

4. Required proof architecture

(1) **Indicator test:** insert a centered indicator and prove $\lambda \le 2\Phi$.
(2) **Discrete coarea:** for $h \ge 0$, prove

$$\sum_{\{x,y\}\in E} |h(x) - h(y)| = \int_0^\infty |\partial_E\{h > t\}| \,\mathrm{d}t.$$

(3) **One-sided estimate:** if $g \ge 0$ and $|\operatorname{supp} g| \le |V|/2$, apply coarea to g^2 and Cauchy–Schwarz to prove

$$\mathcal{E}(g,g) \ge \frac{\Phi^2}{2} \|g\|^2_{L^2(\pi)}.$$

(4) **Median reduction:** subtract a median, decompose into positive and negative parts, justify the support bounds and energy inequality, and conclude the lower Cheeger estimate.

5. Examples required

Discuss the cycle and complete graph. Also construct or describe a regular dumbbell-type graph with two well-connected pieces joined through few cross-edges and explain why the cut predicts slow relaxation.

6. What may be taken as input

The Rayleigh characterization of the spectral gap may be used. The regular-graph proof above should otherwise be self-contained. A general weighted reversible-chain theorem may be stated only after the proof.

7. Exercises and common pitfalls

Possible exercises: estimate Φ of a cycle; prove the easy direction for a reversible chain; or find a bottleneck test function. Do not mix normalized and unnormalized Laplacians. Conductance and edge expansion differ by degree and stationary-mass factors; every constant must be translated to the fixed normalization.

8. Suggested reading route and attribution

(1) D. A. Levin and Y. Peres, with contributions by E. L. Wilmer, *Markov Chains and Mixing Times*, 2nd ed.: Chapter 7, "Lower bounds on mixing times," and Chapter 13, "Eigenfunctions and comparison of chains," including the Cheeger-type inequalities and historical notes (public PDF from the authors).

(2) J. Cheeger, "A lower bound for the smallest eigenvalue of the Laplacian," in R. C. Gunning, ed., *Problems in Analysis*, Princeton University Press, 1970, pp. 195–199. This is the manifold antecedent; graph and reversible-chain versions are attributed in the historical notes of Levin–Peres.

9. Before the required consultation

Bring a completed one-page preparation document containing: the exact theorem statements; a dependency map of the proof; the portion you will prove in full; planned examples; two exercises with solutions; exact source locations; anticipated questions; and any point you do not yet understand. The consultation is for resolving mathematical and expository issues, not for choosing a topic from scratch.

10. AI-use disclosure

The preparation document must disclose any use of generative artificial-intelligence tools. Identify the tool, the task for which it was used, material incorporated into the presentation, and how mathematical assertions and references were independently verified. You remain responsible for every statement in the talk.

This is an instructor-authored assignment guide, not a substitute for reading the listed sources. All normalizations in the talk should agree with this packet unless a change is explicitly approved in advance.