21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY ROUND I PRINCIPAL TOPIC PACKET 03 TENSORIZATION OF VARIANCE AND THE EFRON–STEIN INEQUALITY

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ABSTRACT. **Organizing question:** Why does independence permit coordinatewise variance control? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

AT A GLANCE

Status	Principal Round I topic
Setting	Probability/statistics: functions of independent variables
Primary tools	Conditional variance, Jensen, induction, coordinate resampling
Background	Finite probability; independence; variance
Relative difficulty	2/4
Principal perspective	Probability, statistics, and product structure

1. WHY THIS TOPIC MAY APPEAL TO YOU

This is the most directly statistical topic among the principal assignments. It explains why high dimension need not create large fluctuations when coordinates are independent and develops a proof method that recurs throughout concentration theory.

2. FIXED SETTING

Let $X_1, \dots, X_n$ be independent random variables and let $f(X_1, \dots, X_n)$ be square-integrable. For fixed values of all coordinates except i, write $\mathrm{Var}_i\, f$ for variance in the ith coordinate only.

3. PRINCIPAL THEOREMS

Theorem 1 (Tensorization of variance)**.**

$$\mathrm{Var}(f(X_1, \dots, X_n)) \le \sum_{i=1}^{n} \mathbf{E}[\mathrm{Var}_i\, f(X_1, \dots, X_n)].$$

Let X_i' be an independent copy of X_i and let $X^{(i)}$ replace X_i by X_i'. Deduce

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Theorem 2 (Efron–Stein)**.**

$$\operatorname{Var}(f(X)) \le \frac{1}{2}\sum_{i=1}^{n} \mathbf{E}\big[(f(X) - f(X^{(i)}))^2\big].$$

4. Required proof content

(1) Prove the finite-space law of total variance

$$\operatorname{Var}(Y) = \mathbf{E}[\operatorname{Var}(Y \mid Z)] + \operatorname{Var}(\mathbf{E}[Y \mid Z])$$

by direct algebra.

(2) Prove tensorization first for two coordinates and then by induction, identifying precisely where independence and Jensen's inequality enter.

(3) Prove the conditional resampling identity

$$\mathbf{E}\big[(f(X) - f(X^{(i)}))^2 \mid X_{-i}\big] = 2\operatorname{Var}_i f.$$

(4) Derive Efron–Stein and explain equality for additive functions $f(x) = \sum_i g_i(x_i)$.

5. Required applications

Treat at least two examples: the empirical average; a coordinatewise sensitive statistic; or a Boolean function. For the empirical average, show explicitly how the correct n^{-1} variance scale appears.

6. What may be taken as input

You may work entirely on finite probability spaces and mention the general square-integrable version as an extension. Do not invoke abstract Hilbert-space conditional expectation without explaining the finite calculation it replaces.

7. Exercises and common pitfalls

Possible exercises include equality for additive functions, a product-space Poincare inequality, or a variance estimate for a bounded statistic. Independence is essential; every conditional expectation must display the variables being held fixed; and the factor 1/2 in Efron–Stein must be derived rather than memorized.

8. Suggested reading route and attribution

(1) R. van Handel, *Probability in High Dimension*, Section 2.1, approximately pp. 13–20, for tensorization and bounded differences (public course notes).

(2) B. Efron and C. Stein, "The jackknife estimate of variance," *Annals of Statistics* 9 (1981), no. 3, 586–596. DOI and journal page. This is a historical source; the presentation should use the elementary modern proof above.

9. Before the required consultation

Bring a completed one-page preparation document containing: the exact theorem statements; a dependency map of the proof; the portion you will prove in full; planned examples; two exercises with solutions; exact source locations; anticipated questions; and any point you do not yet understand. The consultation is for resolving mathematical and expository issues, not for choosing a topic from scratch.

10. AI-use disclosure

The preparation document must disclose any use of generative artificial-intelligence tools. Identify the tool, the task for which it was used, material incorporated into the presentation, and how mathematical assertions and references were independently verified. You remain responsible for every statement in the talk.

This is an instructor-authored assignment guide, not a substitute for reading the listed sources. All normalizations in the talk should agree with this packet unless a change is explicitly approved in advance.