21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY ROUND I PRINCIPAL TOPIC PACKET 02 RANDOM WALK ON A CYCLE AND DIFFUSIVE RELAXATION

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Abstract. **Organizing question:** Why does random walk on a cycle relax on the diffusive time scale? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

At a glance

Status	Principal Round I topic
Setting	Discrete: continuous-time simple random walk on $\mathbb{Z}/n\mathbb{Z}$
Primary tools	Finite Fourier analysis, linear algebra, reversible Markov chains
Background	Finite probability; complex numbers; eigenvalues and eigenvectors
Relative difficulty	2/4
Principal perspective	Discrete probability and dynamics

1. Why this topic may appeal to you

This is the discrete counterpart of the first topic. It gives an exact calculation rather than an abstract estimate and shows how geometry, Fourier modes, a spectral gap, and a physical relaxation time encode the same phenomenon.

2. Fixed model and normalization

Let $C_n = \mathbb{Z}/n\mathbb{Z}$ and let π be uniform. Use the continuous-time generator

$$Lf(x) = \frac{1}{2}\big(f(x+1) + f(x-1) - 2f(x)\big).$$

The Dirichlet form is

$$\mathcal{E}(f,f) := -\langle f, Lf\rangle_{L^2(\pi)} = \frac{1}{2n}\sum_{x\in C_n}\big(f(x+1) - f(x)\big)^2.$$

This normalization is fixed.

Date: Fall 2026.

3. Principal theorem

For $k = 0, \dots, n-1$, define $\chi_k(x) = e^{2\pi i k x/n}$. Prove that χ_k is an eigenfunction of $-L$ with eigenvalue

$$\lambda_k = 1 - \cos(2\pi k/n) = 2\sin^2(\pi k/n).$$

Consequently,

$$\lambda_{\mathrm{gap}} = 1 - \cos(2\pi/n) = 2\sin^2(\pi/n) \sim \frac{2\pi^2}{n^2}, \qquad \mathrm{Var}_\pi(f) \le \lambda_{\mathrm{gap}}^{-1}\mathcal{E}(f,f).$$

4. Required proof content

(1) Prove stationarity and reversibility and derive the displayed Dirichlet-form identity.
(2) Establish orthogonality of the finite Fourier characters and diagonalize L completely.
(3) Identify the first nonconstant eigenspace and the equality cases in the Poincaré inequality.
(4) For $P_t = e^{tL}$, prove

$$\mathrm{Var}_\pi(P_t f) \le e^{-2\lambda_{\mathrm{gap}} t}\,\mathrm{Var}_\pi(f).$$

(5) Explain why the characteristic time is of order n^2, connecting the first mode with diffusive spatial scaling.

5. What may be taken as input

You may use the spectral theorem for a real symmetric matrix and $\sin x \sim x$ as $x \to 0$. The Fourier diagonalization itself is required.

6. Examples and exercises

Compare with at least two of: the complete graph, a reflecting path, or a different jump-rate normalization. Prepare two exercises, such as computing C_4 directly, diagonalizing the lazy discrete-time walk, or comparing the first discrete mode with the continuum circle.

7. Common pitfalls

Do not switch between discrete time and continuous time without translating eigenvalues. Keep track of directed versus undirected edge sums. State whether the Fourier calculation is being performed over real- or complex-valued functions.

8. Suggested reading route and attribution

(1) D. A. Levin and Y. Peres, with contributions by E. L. Wilmer, *Markov Chains and Mixing Times*, 2nd ed., AMS, 2017: Chapter 12, "Eigenvalues," and Chapter 20, "Continuous-time chains." Public PDF from the authors.
(2) R. van Handel, *Probability in High Dimension*, APC 550 lecture notes: Sections 2.2–2.4, approximately pp. 21–40, for Markov semigroups, Poincare inequalities, variance identities, and exponential ergodicity (public course notes).

The exact cycle spectrum should be computed in the talk; the references supply the surrounding spectral and semigroup framework.

9. Before the required consultation

Bring a completed one-page preparation document containing: the exact theorem statements; a dependency map of the proof; the portion you will prove in full; planned examples; two exercises with solutions; exact source locations; anticipated questions; and any point you do not yet understand. The consultation is for resolving mathematical and expository issues, not for choosing a topic from scratch.

10. AI-use disclosure

The preparation document must disclose any use of generative artificial-intelligence tools. Identify the tool, the task for which it was used, material incorporated into the presentation, and how mathematical assertions and references were independently verified. You remain responsible for every statement in the talk.

This is an instructor-authored assignment guide, not a substitute for reading the listed sources. All normalizations in the talk should agree with this packet unless a change is explicitly approved in advance.