21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY ROUND I PRINCIPAL TOPIC PACKET 01 SHARP POINCARÉ INEQUALITIES AND BOUNDARY CONDITIONS

MATTHEW ROSENZWEIG

Abstract. **Organizing question:** How do boundary conditions determine the sharp Poincaré constant? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

At a glance

Status	Principal Round I topic
Setting	Continuous: an interval and a circle
Primary tools	Fourier series, eigenfunctions, Rayleigh quotients, integration by parts
Background	Single-variable analysis; elementary ODEs; basic Fourier series helpful
Relative difficulty	2/4
Principal perspective	Analysis/PDE, with spectral and dynamical interpretations

1. Why this topic may appeal to you

This is the most concrete continuous topic in the slate. It starts from calculus and Fourier series, but already exhibits three recurring principles: the null space depends on the model, the best constant is an eigenvalue, and coercivity becomes exponential relaxation under heat flow.

2. Fixed normalization

Let $L > 0$. Work first with real-valued smooth functions and discuss extension to the corresponding H^1 spaces only at the theorem-map level.

(1) **Dirichlet interval:** $f(0) = f(L) = 0$.
(2) **Mean-zero interval:** $f \in C^1([0, L])$ and $\int_0^L f(x)\,\mathrm{d}x = 0$.
(3) **Periodic circle of length L:** f is L-periodic and $\int_0^L f(x)\,\mathrm{d}x = 0$.

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3. Principal theorem

Establish the sharp inequalities

$$\int_0^L |f|^2 \, \mathrm{d}x \le \left(\frac{L}{\pi}\right)^2 \int_0^L |f'|^2 \, \mathrm{d}x \qquad \text{(Dirichlet)},$$
$$\int_0^L |f|^2 \, \mathrm{d}x \le \left(\frac{L}{\pi}\right)^2 \int_0^L |f'|^2 \, \mathrm{d}x \qquad \text{(mean-zero interval)},$$
$$\int_0^L |f|^2 \, \mathrm{d}x \le \left(\frac{L}{2\pi}\right)^2 \int_0^L |f'|^2 \, \mathrm{d}x \qquad \text{(periodic mean-zero)}.$$

Identify the equality modes:

$$\sin(\pi x/L), \qquad \cos(\pi x/L), \qquad \sin(2\pi x/L),\ \cos(2\pi x/L),$$

respectively.

4. Required proof content

(1) Give a complete Fourier/eigenfunction proof in at least two settings. Compute the derivative coefficients and state the Parseval identity being used.
(2) Explain the Rayleigh-quotient interpretation: the reciprocal of the sharp Poincaré constant is the smallest admissible eigenvalue of $-\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ once the zero-energy modes are removed.
(3) Derive the associated heat-flow decay. If $u_t = u_{xx}$ and the relevant equilibrium mode has been removed, show

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^L |u|^2 \, \mathrm{d}x = -2 \int_0^L |u_x|^2 \, \mathrm{d}x \le -2\lambda_1 \int_0^L |u|^2 \, \mathrm{d}x.$$

(4) Explain the differing constants through the lowest admissible frequency, not merely by quoting formulas.

5. What may be taken as input

You may take completeness of the standard sine, cosine, and complex Fourier bases as known. You should not take the sharp constants or equality cases as inputs.

6. Examples and exercises

Include a function proving sharpness in each setting and a function admissible in one setting but not another. Prepare two exercises, for example: rescale from $[0, 1]$ to $[0, L]$; determine the sharp constant in an even or odd subspace; or deduce heat-flow decay.

7. Common pitfalls

Do not conflate a boundary condition with the mean-zero constraint. Do not assert sharpness without an equality mode. When integrating by parts, state exactly why the boundary contribution vanishes.

8. Suggested reading route and attribution

(1) T. H. Colding, notes prepared by D. Winter, *Introduction to Partial Differential Equations*, MIT OpenCourseWare 18.152: Lecture 4, "The Poincare Inequalities," and Lecture 13, "The Heat Equation." The course page identifies the notes as undergraduate and freely available (MIT OpenCourseWare lecture-notes page).

(2) D. Bakry, I. Gentil, and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Springer, 2014, the chapter "Poincare Inequalities," pp. 177–233. DOI and publisher page.

The first source provides undergraduate background. The sharp one-dimensional comparison and exact normalization in this assignment must be reconstructed from the Fourier calculation rather than copied from a general theorem.

9. Before the required consultation

Bring a completed one-page preparation document containing: the exact theorem statements; a dependency map of the proof; the portion you will prove in full; planned examples; two exercises with solutions; exact source locations; anticipated questions; and any point you do not yet understand. The consultation is for resolving mathematical and expository issues, not for choosing a topic from scratch.

10. AI-use disclosure

The preparation document must disclose any use of generative artificial-intelligence tools. Identify the tool, the task for which it was used, material incorporated into the presentation, and how mathematical assertions and references were independently verified. You remain responsible for every statement in the talk.

This is an instructor-authored assignment guide, not a substitute for reading the listed sources. All normalizations in the talk should agree with this packet unless a change is explicitly approved in advance.