21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY ROUND I TOPIC CATALOGUE AND PREFERENCE GUIDE

MATTHEW ROSENZWEIG

Abstract. This catalogue introduces seven principal and three alternate Round I presentation topics. Browse the individual packets before ranking preferences. Final assignments will balance preference, preparation, presentation order, and mathematical coverage.

1. How to use the materials

Round I is a scaffolded first presentation. No topic requires reading an original research paper. Every assignment asks for a canonical model, a precise principal theorem, at least one complete substantive proof, examples, interpretation, two short exercises, source attribution, and an AI-use disclosure.

First read this catalogue. Then read the *At a glance* and *Why this topic may appeal to you* sections of all seven principal packets. Read the complete packets for your likely top choices and browse the alternates if one better matches your interests. Rank according to genuine interest and fit, not only perceived ease.

2. Principal-topic comparison

No.	Organizing question	Setting	Main proof mechanism	Diff.
01	How do boundary conditions determine the sharp Poincaré constant?	Continuous: an interval and a circle	Fourier series, eigenfunctions, Rayleigh quotients, integration by parts	2/4
02	Why does random walk on a cycle relax on the diffusive time scale?	Discrete: continuous-time simple random walk on $\mathbb{Z}/n\mathbb{Z}$	Finite Fourier analysis, linear algebra, reversible Markov chains	2/4
03	Why does independence permit coordinatewise variance control?	Probability/statistics: functions of independent variables	Conditional variance, Jensen, induction, coordinate resampling	2/4
04	How does a bottleneck force slow convergence to equilibrium?	Discrete geometry: finite connected regular graphs	Rayleigh quotients, level sets, discrete coarea, Cauchy–Schwarz	4/4
05	How does a scale-sensitive inequality produce algebraic decay and smoothing?	Continuous/PDE: the heat equation on the real line	One-dimensional calculus, interpolation, energy estimates, scalar ODEs	3/4
06	Why is the Gaussian Poincaré inequality dimension-free?	Continuous probability: standard Gaussian measure on $\mathbb{R}^d$	Mehler formula, Gaussian integration by parts, semigroup interpolation, Jensen	3/4

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07	How does a logarithmic Sobolev inequality produce Gaussian concentration?	Probability measures satisfying a logarithmic Sobolev inequality; Gaussian model	Functional entropy, exponential moments, differentiation, Chernoff optimization	4/4

3. Alternate topics

The alternates may replace a principal topic when they better fit student interest or background.

08. **Boolean cube:** Fourier expansion, Poincare inequality, and coordinate influences.
09. **Pinsker inequality:** relative entropy controls total variation and converts entropy decay into probabilistic convergence.
10. **Entropy, likelihood, and Gibbs equilibrium:** relative entropy as both likelihood loss and free-energy excess.

4. What every Round I talk must contain

(1) A precise statement of the model, including all normalizations.
(2) One principal theorem and at least one complete, substantive proof.
(3) A clear distinction between proved statements and inputs.
(4) Examples showing why the theorem is nontrivial and what the constant means.
(5) An interpretation connected to at least one recurring perspective: analytic/dynamical, physical, statistical/information-theoretic, or expository.
(6) Two short exercises, introduced during the talk and supplied with solutions to the instructor. Class solutions will not be collected or graded.
(7) Exact source attribution and an AI-use disclosure.

5. Preference submission

Submit the Round I Topic Preference Google Form linked in the August 28 course announcement by **Sunday, August 30, 2026, at 11:59 p.m. EDT**. The form asks for five distinct ranked choices, at most one topic you strongly prefer not to receive, relevant mathematical background, and genuine scheduling constraints among the anticipated Round I dates. Assignments are not first-come-first-served. Preferences sent by email will not count as a form submission.

6. Difficulty labels

The labels are relative to Round I and reflect the number of new ideas and length of the proof architecture. They do not measure importance, predict a grade, or cap the quality of a presentation. A lower-difficulty topic can support an excellent and mathematically substantial talk.

7. Topic index

01. **Sharp Poincaré inequalities and boundary conditions.** *How do boundary conditions determine the sharp Poincaré constant?*
02. **Random walk on a cycle and diffusive relaxation.** *Why does random walk on a cycle relax on the diffusive time scale?*

03. **Tensorization of variance and the Efron–Stein inequality.** *Why does independence permit coordinatewise variance control?*
04. **Cheeger inequality and bottlenecks on regular graphs.** *How does a bottleneck force slow convergence to equilibrium?*
05. **The Nash inequality and heat-flow smoothing.** *How does a scale-sensitive inequality produce algebraic decay and smoothing?*
06. **Gaussian Poincaré inequality from the Ornstein–Uhlenbeck semigroup.** *Why is the Gaussian Poincaré inequality dimension-free?*
07. **Logarithmic Sobolev inequalities and the Herbst argument.** *How does a logarithmic Sobolev inequality produce Gaussian concentration?*
08. **Fourier analysis, Poincaré inequality, and influences on the Boolean cube.** *How does Fourier analysis convert coordinate sensitivity into a variance bound?*
09. **Pinsker inequality and entropy-to-total-variation convergence.** *How does relative entropy control probabilistic distinguishability?*
10. **Entropy, likelihood, and Gibbs equilibrium.** *Why does relative entropy appear in both statistical inference and equilibrium statistical mechanics?*

21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY ROUND I PRINCIPAL TOPIC PACKET 01 SHARP POINCARÉ INEQUALITIES AND BOUNDARY CONDITIONS

MATTHEW ROSENZWEIG

ABSTRACT. **Organizing question:** How do boundary conditions determine the sharp Poincaré constant? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

AT A GLANCE

Status	Principal Round I topic
Setting	Continuous: an interval and a circle
Primary tools	Fourier series, eigenfunctions, Rayleigh quotients, integration by parts
Background	Single-variable analysis; elementary ODEs; basic Fourier series helpful
Relative difficulty	2/4
Principal perspective	Analysis/PDE, with spectral and dynamical interpretations

1. WHY THIS TOPIC MAY APPEAL TO YOU

This is the most concrete continuous topic in the slate. It starts from calculus and Fourier series, but already exhibits three recurring principles: the null space depends on the model, the best constant is an eigenvalue, and coercivity becomes exponential relaxation under heat flow.

2. FIXED NORMALIZATION

Let $L > 0$. Work first with real-valued smooth functions and discuss extension to the corresponding H^1 spaces only at the theorem-map level.

(1) **Dirichlet interval:** $f(0) = f(L) = 0$.
(2) **Mean-zero interval:** $f \in C^1([0, L])$ and $\int_0^L f(x)\, \mathrm{d}x = 0$.
(3) **Periodic circle of length L:** f is L-periodic and $\int_0^L f(x)\, \mathrm{d}x = 0$.

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21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY ROUND I PRINCIPAL TOPIC PACKET 02 RANDOM WALK ON A CYCLE AND DIFFUSIVE RELAXATION

MATTHEW ROSENZWEIG

ABSTRACT. **Organizing question:** Why does random walk on a cycle relax on the diffusive time scale? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

At a glance

Status	Principal Round I topic
Setting	Discrete: continuous-time simple random walk on $\mathbb{Z}/n\mathbb{Z}$
Primary tools	Finite Fourier analysis, linear algebra, reversible Markov chains
Background	Finite probability; complex numbers; eigenvalues and eigenvectors
Relative difficulty	2/4
Principal perspective	Discrete probability and dynamics

1. Why this topic may appeal to you

This is the discrete counterpart of the first topic. It gives an exact calculation rather than an abstract estimate and shows how geometry, Fourier modes, a spectral gap, and a physical relaxation time encode the same phenomenon.

2. Fixed model and normalization

Let $C_n = \mathbb{Z}/n\mathbb{Z}$ and let π be uniform. Use the continuous-time generator

$$Lf(x) = \frac{1}{2}\big(f(x+1) + f(x-1) - 2f(x)\big).$$

The Dirichlet form is

$$\mathcal{E}(f,f) := -\langle f, Lf\rangle_{L^2(\pi)} = \frac{1}{2n}\sum_{x\in C_n}\big(f(x+1) - f(x)\big)^2.$$

This normalization is fixed.

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21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY ROUND I PRINCIPAL TOPIC PACKET 03 TENSORIZATION OF VARIANCE AND THE EFRON–STEIN INEQUALITY

MATTHEW ROSENZWEIG

Abstract. **Organizing question:** Why does independence permit coordinatewise variance control? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

At a glance

Status	Principal Round I topic
Setting	Probability/statistics: functions of independent variables
Primary tools	Conditional variance, Jensen, induction, coordinate resampling
Background	Finite probability; independence; variance
Relative difficulty	2/4
Principal perspective	Probability, statistics, and product structure

1. Why this topic may appeal to you

This is the most directly statistical topic among the principal assignments. It explains why high dimension need not create large fluctuations when coordinates are independent and develops a proof method that recurs throughout concentration theory.

2. Fixed setting

Let $X_1, \ldots, X_n$ be independent random variables and let $f(X_1, \ldots, X_n)$ be square-integrable. For fixed values of all coordinates except i, write $\mathrm{Var}_i\, f$ for variance in the ith coordinate only.

3. Principal theorems

Theorem 1 (Tensorization of variance)**.**

$$\mathrm{Var}(f(X_1, \ldots, X_n)) \leq \sum_{i=1}^{n} \mathbf{E}[\mathrm{Var}_i\, f(X_1, \ldots, X_n)].$$

Let X_i' be an independent copy of X_i and let $X^{(i)}$ replace X_i by X_i'. Deduce

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CHEEGER INEQUALITY AND BOTTLENECKS ON REGULAR GRAPHS

MATTHEW ROSENZWEIG

Abstract. **Organizing question:** How does a bottleneck force slow convergence to equilibrium? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

At a glance

Status	Principal Round I topic
Setting	Discrete geometry: finite connected regular graphs
Primary tools	Rayleigh quotients, level sets, discrete coarea, Cauchy–Schwarz
Background	Graph theory; finite Markov chains; comfort with multi-step proofs
Relative difficulty	4/4
Principal perspective	Discrete geometry, probability, and spectral theory

1. Why this topic may appeal to you

This is the most geometric discrete topic and one of the most demanding Round I choices. It converts a global analytic quantity, the spectral gap, into a statement about cuts and bottlenecks. The proof contains a genuine mechanism rather than only an exact calculation.

2. Fixed normalization

Let $G = (V, E)$ be a finite connected d-regular graph. Let P be the simple-random-walk transition matrix, $L = P - I$, and π uniform. Then

$$\mathcal{E}(f, f) = \frac{1}{d|V|} \sum_{\{x,y\} \in E} (f(x) - f(y))^2.$$

Define

$$\Phi := \min_{\varnothing \neq S \subseteq V,\ |S| \leq |V|/2} \frac{|\partial_E S|}{d|S|},$$

where $\partial_E S$ consists of edges with exactly one endpoint in S. Let λ be the spectral gap of $I - P$.

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21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY ROUND I PRINCIPAL TOPIC PACKET 05 THE NASH INEQUALITY AND HEAT-FLOW SMOOTHING

MATTHEW ROSENZWEIG

Abstract. **Organizing question:** How does a scale-sensitive inequality produce algebraic decay and smoothing? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

At a glance

Status	Principal Round I topic
Setting	Continuous/PDE: the heat equation on the real line
Primary tools	One-dimensional calculus, interpolation, energy estimates, scalar ODEs
Background	Single-variable analysis; elementary PDE notation
Relative difficulty	3/4
Principal perspective	Analysis/PDE and scale-sensitive functional inequalities

1. Why this topic may appeal to you

This topic is the cleanest way to see the branch of the course that is not organized around an equilibrium probability measure. It explains why whole-space heat flow decays algebraically rather than exponentially and why scaling determines the exponents.

2. Fixed setting and principal inequalities

Work first with $f \in C_c^\infty(\mathbb{R})$. Prove

$$\|f\|_\infty^2 \le 2\,\|f\|_2\,\|f'\|_2$$

and deduce the one-dimensional Nash inequality

$$\|f\|_2^6 \le 4\,\|f\|_1^4\,\|f'\|_2^2.$$

The constant is secondary; the exponents and scaling are essential.

3. Required heat-flow application

For $u_t = u_{xx}$ on $\mathbb{R}$, set $y(t) = \|u(t)\|_2^2$ and $M = \|u(0)\|_1$. Using L^1 contraction and the energy identity, prove

$$y'(t) = -2\,\|u_x(t)\|_2^2 \le -\frac{y(t)^3}{2M^4}.$$

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21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY ROUND I PRINCIPAL TOPIC PACKET 06 GAUSSIAN POINCARÉ INEQUALITY FROM THE ORNSTEIN–UHLENBECK SEMIGROUP

MATTHEW ROSENZWEIG

ABSTRACT. **Organizing question:** Why is the Gaussian Poincaré inequality dimension-free? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

AT A GLANCE

Status	Principal Round I topic
Setting	Continuous probability: standard Gaussian measure on $\mathbb{R}^d$
Primary tools	Mehler formula, Gaussian integration by parts, semigroup interpolation, Jensen
Background	Multivariable calculus; Gaussian random vectors; basic semigroup language
Relative difficulty	3/4
Principal perspective	Probability, dynamics, and high-dimensional analysis

1. WHY THIS TOPIC MAY APPEAL TO YOU

This is the first genuinely high-dimensional continuous model. It gives a complete semigroup proof, exhibits several reusable identities, and explains why independent Gaussian coordinates do not force deterioration of the Poincare constant.

2. FIXED NORMALIZATION

Let γ_d be standard Gaussian measure on $\mathbb{R}^d$ and define

$$P_t f(x) = \mathbf{E}\left[f\left(e^{-t}x + \sqrt{1-e^{-2t}}\,G\right)\right], \qquad G \sim \gamma_d.$$

The generator is $L = \Delta - x \cdot \nabla$.

3. PRINCIPAL THEOREM

For smooth f with sufficient growth control,

$$\mathrm{Var}_{\gamma_d}(f) \le \int_{\mathbb{R}^d} |\nabla f|^2 \,\mathrm{d}\gamma_d.$$

The constant 1 is sharp, independent of d, and attained by nonconstant affine functions.

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21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY ROUND I PRINCIPAL TOPIC PACKET 07 LOGARITHMIC SOBOLEV INEQUALITIES AND THE HERBST ARGUMENT

MATTHEW ROSENZWEIG

ABSTRACT. **Organizing question:** How does a logarithmic Sobolev inequality produce Gaussian concentration? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

At a glance

Status	Principal Round I topic
Setting	Probability measures satisfying a logarithmic Sobolev inequality; Gaussian model
Primary tools	Functional entropy, exponential moments, differentiation, Chernoff optimization
Background	Calculus; probability; comfort manipulating expectations
Relative difficulty	4/4
Principal perspective	Information theory, probability, and concentration

1. Why this topic may appeal to you

This topic gives one of the central implication chains in modern probability: a local energy inequality controls exponential moments and hence rare fluctuations. It has the strongest information-theoretic flavor among the principal topics.

2. Fixed functional inequality

Assume a probability measure μ on $\mathbb{R}^d$ satisfies

$$\mathrm{Ent}_\mu(e^g) \leq \frac{C}{2} \int e^g |\nabla g|^2 \,\mathrm{d}\mu$$

for smooth g, where

$$\mathrm{Ent}_\mu(h) = \int h \log h \,\mathrm{d}\mu - \left(\int h \,\mathrm{d}\mu\right) \log \left(\int h \,\mathrm{d}\mu\right).$$

For standard Gaussian measure, $C = 1$ in this normalization.

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21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY ROUND I ALTERNATE TOPIC PACKET 08 FOURIER ANALYSIS, POINCARÉ INEQUALITY, AND INFLUENCES ON THE BOOLEAN CUBE

MATTHEW ROSENZWEIG

ABSTRACT. **Organizing question:** How does Fourier analysis convert coordinate sensitivity into a variance bound? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

AT A GLANCE

Status	Alternate Round I topic
Setting	Discrete product space $\{-1,1\}^n$
Primary tools	Walsh–Fourier basis, Parseval, discrete derivatives, influences
Background	Linear algebra; finite probability; elementary combinatorics
Relative difficulty	3/4
Principal perspective	Discrete probability and theoretical computer science

1. WHY THIS TOPIC MAY APPEAL TO YOU

Choose this alternate if you are interested in discrete mathematics, theoretical computer science, learning theory, or Boolean functions. It gives a finite-dimensional Fourier theory parallel to the cycle calculation but emphasizes coordinate sensitivity rather than geometry in physical space.

2. FIXED SETTING

Let $\Omega = \{-1,1\}^n$ with uniform measure. For $S \subseteq [n]$, define

$$\chi_S(x) = \prod_{i\in S} x_i, \qquad D_i f(x) = \frac{f(x) - f(x^{\oplus i})}{2},$$

where $x^{\oplus i}$ flips coordinate i.

3. PRINCIPAL THEOREM

Develop

$$f = \sum_{S\subseteq[n]} \widehat{f}(S)\chi_S$$

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21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY ROUND I ALTERNATE TOPIC PACKET 09 PINSKER INEQUALITY AND ENTROPY-TO-TOTAL-VARIATION CONVERGENCE

MATTHEW ROSENZWEIG

ABSTRACT. **Organizing question:** How does relative entropy control probabilistic distinguishability? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

At a glance

Status	Alternate Round I topic
Setting	Information theory on finite probability spaces
Primary tools	Relative entropy, log-sum inequality, coarse-graining, convexity
Background	Finite probability; single-variable calculus
Relative difficulty	2/4
Principal perspective	Information theory and convergence to equilibrium

1. Why this topic may appeal to you

This alternate is a focused information-theoretic topic with a complete elementary proof. It explains how an analytic entropy estimate becomes an operational probabilistic statement about events and mixing.

2. Fixed conventions

For probability measures ν, μ on a finite set with $\mu(x) > 0$, use natural logarithms and define

$$D(\nu\|\mu) = \sum_x \nu(x) \log \frac{\nu(x)}{\mu(x)}, \qquad \|\nu - \mu\|_{\mathrm{TV}} = \frac{1}{2} \sum_x |\nu(x) - \mu(x)|.$$

3. Principal theorem

Prove Pinsker's inequality

$$\|\nu - \mu\|_{\mathrm{TV}} \le \sqrt{\frac{1}{2} D(\nu\|\mu)}.$$

Then deduce from

$$D(\nu_t\|\mu) \le e^{-\alpha t} D(\nu_0\|\mu)$$

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ENTROPY, LIKELIHOOD, AND GIBBS EQUILIBRIUM

MATTHEW ROSENZWEIG

Abstract. **Organizing question:** Why does relative entropy appear in both statistical inference and equilibrium statistical mechanics? This packet fixes the mathematical scope and normalization, identifies the required proof content, and provides a curated reading route. It is intended both for informed preference ranking and for preparation after assignment.

At a glance

Status	Alternate Round I topic
Setting	Finite probability spaces
Primary tools	Jensen, relative entropy, likelihood identities, variational principles
Background	Finite probability; logarithms; elementary optimization
Relative difficulty	2/4
Principal perspective	Statistical/information-theoretic and physical

1. Why this topic may appeal to you

This alternate is the most conceptual bridge between statistics and physics. It is not a broad historical survey: the talk is organized around two exact identities showing that the same relative entropy measures likelihood loss and free-energy excess.

2. Statistical identity

Let $x_1, \dots, x_n$ lie in a finite alphabet, let $\widehat{p}_n$ be the empirical law, and let q be positive on the observed symbols. Prove

$$\frac{1}{n} \log \prod_{j=1}^{n} q(x_j) = \sum_x \widehat{p}_n(x) \log q(x) = -H(\widehat{p}_n) - D(\widehat{p}_n \| q),$$

where $H(p) = -\sum_x p(x) \log p(x)$. Conclude that the maximum-likelihood estimate over all probability laws is $q = \widehat{p}_n$. Also prove

$$\mathbf{E}_p \left[\log \frac{q(X)}{p(X)} \right] = -D(p \| q).$$

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21-317 FUNCTIONAL INEQUALITIES: FROM ANALYSIS TO PROBABILITY ROUND I ONLINE PREFERENCE SUBMISSION

MATTHEW ROSENZWEIG

Deadline: Sunday, August 30, 2026, at 11:59 p.m. EDT

Submit your ranked Round I topic preferences through the Google Form linked in the August 28 course announcement. The announcement is also accessible from the course website. Preferences sent by email will not count as a form submission.

Before submitting, read the topic catalogue, browse the first-page overview of every principal packet, and read the complete packets for your likely top choices. The form asks you to provide:

- five distinct ranked topic choices;
- at most one topic you strongly prefer not to receive;
- relevant mathematical background and areas of interest;
- genuine scheduling conflicts among the anticipated Round I dates; and
- your current presentation confidence and any especially useful form of instructor support.

Assignments are **not first-come-first-served**. They will balance student preferences, preparation, presentation order, and mathematical coverage. You may edit a submitted response until the deadline using the edit-response link supplied by Google Forms.

The anticipated Round I dates are September 14, 16, 18, 21, 23, 25, and 28. List only genuine conflicts.

If a technical problem prevents you from accessing or submitting the form, contact the instructor before the deadline. As required by the course email policy, begin the subject line with `[21-317]`.

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