Three ways of writing one proof passage

The general $O(n)$ model of Bauerschmidt–Bodineau

Classroom comparison packet

Shared mathematical setup (read once)

Let Λ be finite and let each spin take values in $S^{n-1} \subset \mathbb{R}^n$ (for $n = 1$, the two-point space $S^0 = \{-1, 1\}$). Write μ for normalized surface measure. The Gaussian representation of Bauerschmidt–Bodineau, *A very simple proof of the LSI for high temperature spin systems*, J. Funct. Anal. **276** (2019), 2582–2588, doi:10.1016/j.jfa.2019.01.007. expresses the interacting Gibbs measure as

$$\nu(H) = \nu_{\mathrm{r}}\big(\mu_\varphi(H)\big), \qquad \mu_\varphi = \bigotimes_{x\in\Lambda} \mu_{\varphi_x}, \qquad \mu_{\varphi_x}(ds) = \frac{e^{c\varphi_x\cdot s}}{\int e^{c\varphi_x\cdot t}\,\mu(dt)}\,\mu(ds).$$

Here $c > 0$ is the Gaussian parameter. The conditional laws satisfy, uniformly in φ_x, the one-spin logarithmic Sobolev inequality with constant $\gamma > 0$, hence also the Poincaré inequality

$$\mathrm{Var}_{\mu_{\varphi_x}}(f) \le \gamma^{-1}\mu_{\varphi_x}(|\nabla f|^2).$$

For $n \ge 2$, ∇ is the spherical gradient; for $n = 1$, $|\nabla f(s)|^2 = |f(s) - f(-s)|^2$. We write $\nabla_{\sigma_x} F$ for the corresponding one-site gradient. The law ν_{r} satisfies an auxiliary-field LSI, with constant $\lambda > 0$, so entropy decomposition yields

$$\mathrm{Ent}_\nu(F^2) \le \frac{2}{\gamma}\sum_x \nu(|\nabla_{\sigma_x} F|^2) + \frac{2}{\lambda}\sum_x \nu_{\mathrm{r}}(|\nabla_{\varphi_x} G|^2), \qquad G(\varphi) = \mu_\varphi(F^2)^{1/2}. \tag{S}$$

The precise task is to prove, for every site x, the *gradient-transfer estimate*

$$|\nabla_{\varphi_x} G(\varphi)|^2 \le \frac{2c^2}{\gamma}\,\mu_\varphi(|\nabla_{\sigma_x} F|^2), \tag{T}$$

which bounds the remaining auxiliary-field term by the same spin-gradient energy as the first term in (S). This is BB's equation (15); the steps are in equations (11)–(15).

Fixed writing assignment. Explain the proof of (T), with the intended readers being mathematically prepared undergraduates who know what a logarithmic Sobolev inequality is but have not read BB. Explain why (T) is needed, and distinguish calculations proved here from imported results.

Comparison. The three specimens below all use the same theorem, hypotheses, and calculation. They differ in *organization, explanation, and audience*. The labels are teaching judgments, not objective ranks for every audience.

Bauerschmidt–Bodineau, *A very simple proof of the LSI for high temperature spin systems*, J. Funct. Anal. **276** (2019), 2582–2588, doi:10.1016/j.jfa.2019.01.007. The proof is in equations (9)–(15). The uniform single-spin LSI is quoted from the paper; no new LSI is proved in these specimens.

Version 2: OK

A correct but relatively compressed treatment

Bauerschmidt and Bodineau prove a uniform logarithmic Sobolev inequality for the interacting $O(n)$ model. Their argument introduces an auxiliary Gaussian field φ such that the spins are independent conditional on φ. We now give an estimate used toward the end of their proof.

With $G(\varphi) = \mu_\varphi(F^2)^{1/2}$, differentiating the normalized expectation gives

$$\nabla_{\varphi_x} G = \frac{c}{2\mu_\varphi(F^2)^{1/2}} \operatorname{Cov}_{\mu_\varphi}(F^2, \sigma_x). \tag{O1}$$

The covariance is a vector in $\mathbb{R}^n$. Assume F is not identically zero; otherwise the desired bound is trivial.

Fix the spins other than x and write $f(s) = F(\sigma_{\neq x}, s)$. If S, S' are independent with law μ_{φ_x}, then

$$\operatorname{Cov}(f(S)^2, S) = \tfrac{1}{2}\mathbb{E}\big[(f(S) - f(S'))(f(S) + f(S'))(S - S')\big].$$

By Cauchy–Schwarz, $|S - S'| \le 2$, and $\mathbb{E}(f(S) - f(S'))^2 = 2\operatorname{Var}(f(S))$, we obtain

$$|\operatorname{Cov}_{\mu_{\varphi_x}}(f^2, S)|^2 \le 8 \operatorname{Var}_{\mu_{\varphi_x}}(f)\, \mu_{\varphi_x}(f^2). \tag{O2}$$

Averaging over the remaining spins, using conditional independence and applying Cauchy–Schwarz a second time, gives

$$|\operatorname{Cov}_{\mu_\varphi}(F^2, \sigma_x)|^2 \le 8\mu_\varphi(\operatorname{Var}_{\mu_{\varphi_x}} F)\, \mu_\varphi(F^2).$$

The one-spin Poincaré inequality yields

$$\mu_\varphi(\operatorname{Var}_{\mu_{\varphi_x}} F) \le \gamma^{-1}\mu_\varphi(|\nabla_{\sigma_x} F|^2).$$

Combining these estimates with (O1), we conclude that

$$|\nabla_{\varphi_x} G|^2 \le \frac{2c^2}{\gamma}\, \mu_\varphi(|\nabla_{\sigma_x} F|^2).$$

This proves the derivative bound, which can be inserted into the entropy inequality. The full argument and other inputs appear in Bauerschmidt–Bodineau, equations (5)–(15).

Version 3: Excellent

The same proof, with mathematical purpose made explicit

The Gaussian representation in Bauerschmidt–Bodineau writes the interacting $O(n)$ Gibbs measure as a *mixture* of product measures, not as a product measure itself. This distinction matters after entropy decomposition: tensorization controls the conditional entropy, and the auxiliary-field LSI controls the entropy of $G(\varphi)^2$, but leaves the field derivative in the second term of (S). To complete the theorem, we must bound this derivative using *spin* gradients. We establish precisely (T).

For $S \in S^{n-1}$, the conditional one-spin law is proportional to $e^{c\varphi_x \cdot S}\mu(dS)$. The normalizing factor depends on φ_x, so differentiating gives a *covariance* rather than an uncentered moment:

$$\nabla_{\varphi_x} G = \frac{c}{2\mu_\varphi(F^2)^{1/2}} \operatorname{Cov}_{\mu_\varphi}(F^2, \sigma_x). \tag{E1}$$

For nonzero F, the denominator is positive because all conditional product laws have the same support. The task is now to convert a covariance of F^2 into a quadratic expression involving differences of F.

Hold the spins away from x fixed; write $f(s) = F(\sigma_{\neq x}, s)$ and take independent S, S' with law μ_{φ_x}. The duplication identity gives

$$\operatorname{Cov}(f(S)^2, S) = \tfrac{1}{2}\mathbb{E}\big[(f(S) - f(S'))(f(S) + f(S'))(S - S')\big].$$

This identity is useful because it introduces the difference $f(S) - f(S')$, which vanishes for constant f and can therefore be controlled by a Poincaré inequality. Since $|S - S'| \le 2$ on the unit sphere, Cauchy–Schwarz, together with

$$\mathbb{E}(f(S) - f(S'))^2 = 2\operatorname{Var}(f(S)), \qquad \mathbb{E}(f(S) + f(S'))^2 \le 4\mathbb{E}f(S)^2,$$

implies

$$|\operatorname{Cov}_{\mu_{\varphi_x}}(f^2, S)|^2 \le 8 \operatorname{Var}_{\mu_{\varphi_x}}(f)\, \mu_{\varphi_x}(f^2). \tag{E2}$$

Because μ_φ is a conditional product law, its full covariance is the average of the one-site covariances: the conditional mean of σ_x does not depend on the other spins. Averaging (E2) and applying Cauchy–Schwarz *across those other spins* yields

$$|\operatorname{Cov}_{\mu_\varphi}(F^2, \sigma_x)|^2 \le 8\mu_\varphi(\operatorname{Var}_{\mu_{\varphi_x}} F)\, \mu_\varphi(F^2).$$

The uniform one-spin LSI supplies a Poincaré inequality with constant γ^{-1}, so the first factor is bounded by $\gamma^{-1}\mu_\varphi(|\nabla_{\sigma_x} F|^2)$. Substitution into (E1) proves (T).

Integrating (T) against ν_{r} controls the second term of (S) by the *same* spin Dirichlet form as the first. Thus the covariance computation is not an isolated technical step: it is what closes the entropy estimate for the original interacting Gibbs measure. The spectral hypothesis on the interaction matrix enters in obtaining the auxiliary-field LSI, not in this local covariance bound.

Version 4: Professional

A reusable claim and a more economical proof

The Gaussian representation reduces the interacting-spin logarithmic Sobolev inequality to an entropy decomposition involving conditional spin measures and an auxiliary-field marginal. The final transfer estimate is logically independent of the spectral condition on the interaction matrix: that condition is used to obtain the auxiliary-field LSI. We therefore isolate the local calculation.

Claim. Let $\mu_\varphi = \bigotimes_{x\in\Lambda} \mu_{\varphi_x}$, where $\mu_{\varphi_x}(ds) \propto e^{c\varphi_x\cdot s}\mu(ds)$ on S^{n-1}, and assume the uniform one-spin Poincaré bound $\mathrm{Var}_{\mu_{\varphi_x}} f \le \gamma^{-1}\mu_{\varphi_x}(|\nabla f|^2)$. For a real-valued F set $G(\varphi) = \mu_\varphi(F^2)^{1/2}$. Then

$$|\nabla_{\varphi_x} G(\varphi)|^2 \le \frac{2c^2}{\gamma}\mu_\varphi(|\nabla_{\sigma_x} F|^2), \qquad x \in \Lambda. \tag{P1}$$

Proof. If $F \equiv 0$, the claim is immediate. Otherwise normalized differentiation gives

$$2G\,\nabla_{\varphi_x} G = c\,\mathrm{Cov}_{\mu_\varphi}(F^2, \sigma_x).$$

Condition on all spins except σ_x. For independent S, S' with law μ_{φ_x}, put $f(s) = F(\sigma_{\neq x}, s)$. The duplication identity

$$2\,\mathrm{Cov}(f(S)^2, S) = \mathbb{E}\big[(f(S) - f(S'))(f(S) + f(S'))(S - S')\big]$$

and the bounds $|S - S'| \le 2$ and $\mathbb{E}(f(S) + f(S'))^2 \le 4\mathbb{E}f(S)^2$ give

$$|\,\mathrm{Cov}_{\mu_{\varphi_x}}(f^2, S)|^2 \le 8\,\mathrm{Var}_{\mu_{\varphi_x}}(f)\mu_{\varphi_x}(f^2).$$

Conditional independence and Cauchy–Schwarz, followed by the one-spin Poincaré inequality, imply

$$\begin{aligned} |\,\mathrm{Cov}_{\mu_\varphi}(F^2, \sigma_x)|^2 &\le 8\mu_\varphi(\mathrm{Var}_{\mu_{\varphi_x}} F)\mu_\varphi(F^2) \\ &\le \frac{8}{\gamma}\mu_\varphi(|\nabla_{\sigma_x} F|^2)\mu_\varphi(F^2). \end{aligned}$$

Division by $4G^2 = 4\mu_\varphi(F^2)$ proves (P1). □

Application to BB. Combining (P1) with entropy decomposition (S) gives

$$\mathrm{Ent}_\nu(F^2) \le \frac{2}{\gamma}\left(1 + \frac{2c^2}{\lambda}\right)\sum_x \nu(|\nabla_{\sigma_x} F|^2).$$

In the $O(n)$ model, $\lambda = c - c^2/n > 0$ when $0 < c < n$. For a positive-definite coupling representative with largest eigenvalue strictly less than n, one chooses c strictly between that eigenvalue and n; taking c down to that eigenvalue recovers BB's constant. The claim itself only needs the bounded-spin covariance calculation and a one-spin Poincaré inequality; for unbounded spins, the use of $|S - S'| \le 2$ cannot be retained unchanged. Compare BB, Section 2(iii).