

21-317 MEETING 14: WEAK DERIVATIVES AND THE EUCLIDEAN BRANCH H^1 , SCALING, SOBOLEV, NASH, AND HEAT-FLOW SMOOTHING

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ABSTRACT. Meeting 13 ended immediately after the definition of weak derivative. These notes begin by checking that definition for $u(x) = |x|$, introduce $W^{1,p}$ and H^1 , explain smooth approximation and zero modes, and derive the scale-critical exponent in the Euclidean Sobolev inequality. Nash's inequality and heat-flow smoothing are included as continuation material. The principal goal is to complete the common continuous-analysis vocabulary before the student presentations begin.

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RECAP: WEAK DERIVATIVE

A function $v_i \in L^1_{\text{loc}}(\Omega)$ is the weak derivative $\partial_i u$ of $u \in L^1_{\text{loc}}(\Omega)$ if

$$(0.1) \quad \int_{\Omega} u \partial_i \varphi \, dx = - \int_{\Omega} v_i \varphi \, dx$$

for every $\varphi \in C_c^\infty(\Omega)$.

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19. WEAK DERIVATIVES, CONTINUED

Proposition 19.1. *On $(-1, 1)$, $u(x) = |x|$ has weak derivative*

$$\operatorname{sgn}(x) = \begin{cases} -1, & x < 0, \\ 1, & x > 0. \end{cases}$$

Proof. For $\varphi \in C_c^\infty(-1, 1)$,

$$\begin{aligned} \int_{-1}^1 |x| \varphi'(x) \, dx &= \int_{-1}^0 (-x) \varphi'(x) \, dx + \int_0^1 x \varphi'(x) \, dx \\ &= \int_{-1}^0 \varphi(x) \, dx - \int_0^1 \varphi(x) \, dx \\ &= - \int_{-1}^1 \operatorname{sgn}(x) \varphi(x) \, dx. \end{aligned}$$

□

The step function $H = \mathbf{1}_{(0,1)}$ does not have an L^1 weak derivative: formally its derivative is the point mass δ_0 . A corner may have a function-valued weak derivative, whereas a jump creates a singular derivative.

20. SOBOLEV SPACES

Definition 20.1. For $1 \leq p \leq \infty$,

$$(20.1) \quad W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \partial_i u \in L^p(\Omega), \, 1 \leq i \leq d\}.$$

The space $H^1(\Omega) = W^{1,2}(\Omega)$ has norm

$$(20.2) \quad \|u\|_{H^1}^2 = \|u\|_2^2 + \|\nabla u\|_2^2.$$

The gradient term is the continuous local energy. Smooth functions are useful for computations; H^1 is the natural closure of finite-energy functions.

21. MOLLIFICATION, DENSITY, AND ZERO MODES

For a smooth mollifier η_ε ,

$$u_\varepsilon = u * \eta_\varepsilon$$

is smooth and, under standard hypotheses,

$$u_\varepsilon \rightarrow u, \quad \nabla u_\varepsilon \rightarrow \nabla u$$

in the relevant L^p norms. Thus one often proves a functional inequality for smooth functions and extends it to H^1 by approximation.

Constants have zero gradient. Therefore:

- Dirichlet boundary conditions remove nonzero constants;
- periodic or Neumann problems require subtracting the mean;
- disconnected domains have one constant zero mode on each component.

22. SCALING AND EUCLIDEAN SOBOLEV

For $u_a(x) = u(ax)$,

$$(22.1) \quad \|u_a\|_q = a^{-d/q} \|u\|_q,$$

$$(22.2) \quad \|\nabla u_a\|_2 = a^{1-d/2} \|\nabla u\|_2.$$

Scale invariance forces

$$(22.3) \quad q = 2^* = \frac{2d}{d-2}.$$

Theorem 22.1 (Euclidean Sobolev inequality). *For $d \geq 3$,*

$$(22.4) \quad \|u\|_{2^*} \leq S_d \|\nabla u\|_2, \quad u \in C_c^\infty(\mathbb{R}^d).$$

The geometric proof proceeds through level sets, the Euclidean isoperimetric inequality, and the coarea formula.

23. NASH INEQUALITY

Theorem 23.1. *For $d \geq 3$,*

$$(23.1) \quad \|u\|_2^{2+4/d} \leq C_d \|u\|_1^{4/d} \|\nabla u\|_2^2.$$

Proof. Since

$$\frac{1}{2} = \frac{2}{d+2} \cdot 1 + \frac{d}{d+2} \cdot \frac{1}{2^*},$$

interpolation gives

$$\|u\|_2 \leq \|u\|_1^{2/(d+2)} \|u\|_{2^*}^{d/(d+2)}.$$

Apply (22.4) and raise to the power $2(d+2)/d$. □

24. HEAT-FLOW SMOOTHING

Let $u_t = e^{t\Delta} u_0$. Then

$$(24.1) \quad \frac{d}{dt} \|u_t\|_2^2 = -2 \|\nabla u_t\|_2^2$$

and

$$\|u_t\|_1 \leq \|u_0\|_1 =: M.$$

Set $Y(t) = \|u_t\|_2^2$. Nash gives

$$Y'(t) \leq -c_d M^{-4/d} Y(t)^{1+2/d}.$$

Solving yields

$$(24.2) \quad \boxed{\|e^{t\Delta} u_0\|_2 \leq C_d t^{-d/4} \|u_0\|_1}.$$

The stronger $L^1 \rightarrow L^\infty$ consequence has rate $t^{-d/2}$.

25. TWO COERCIVITY MECHANISMS

	Confined probability space	Euclidean space
Inequality	Poincaré / LSI	Sobolev / Nash
Consequence	exponential relaxation	algebraic smoothing
Geometry	fixed equilibrium scale	dilation symmetry

EXERCISES

Exercise 25.1. Verify directly that the weak derivative of $|x|$ is $\text{sgn}(x)$.

Exercise 25.2. Derive (22.3) from (22.1)–(22.2).

Exercise 25.3. Derive Nash’s inequality from Sobolev by interpolation.

Exercise 25.4. Solve the scalar differential inequality for $Y(t)$ while retaining the initial term.

FURTHER READING

H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*; L. C. Evans, *Partial Differential Equations*, Chapter 5; E. H. Lieb and M. Loss, *Analysis*; J. Nash, “Continuity of solutions of parabolic and elliptic equations,” 1958.