

21-317 MEETING 13: FROM INTEGRATION TO WEAK DERIVATIVES REVISED STUDENT NOTES AND EXERCISES

MATTHEW ROSENZWEIG

ABSTRACT. Meeting 13 recalled the convergence theorems, showed by a moving-spike example that pointwise convergence alone does not control integrals, completed the integration toolkit through product measures and Radon–Nikodym densities, and motivated weak differentiation by integration by parts. The class ended immediately after the definition of weak derivative. Examples and Sobolev spaces begin in Meeting 14.

CONTENTS

Convergence recap and warning	1
18. Operational integration toolkit, continued	1
18.3. Product measures and iterated integration	1
18.4. Absolute continuity and densities	2
19. Weak derivatives	2
Exercises	2
Further reading	2

CONVERGENCE RECAP AND WARNING

Monotone convergence, Fatou’s lemma, and dominated convergence provide different ways to pass a limit under an integral. Pointwise convergence alone is insufficient: on $(0, 1)$,

$$(0.1) \quad f_n(x) = n\mathbf{1}_{(0,1/n)}(x)$$

converges pointwise to zero for every $x > 0$, but

$$\int_0^1 f_n(x) \, dx = 1.$$

18. OPERATIONAL INTEGRATION TOOLKIT, CONTINUED

18.3. Product measures and iterated integration. For sigma-finite measure spaces $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$, the product measure satisfies

$$(\mu \otimes \nu)(A \times B) = \mu(A)\nu(B).$$

Date: September 23, 2026.

If $F \geq 0$ is measurable, Tonelli permits iterated integration, possibly with value $+\infty$. If $F \in L^1(\mu \otimes \nu)$, Fubini gives

$$\int_{X \times Y} F \, d(\mu \otimes \nu) = \int_X \int_Y F(x, y) \, d\nu(y) \, d\mu(x) = \int_Y \int_X F(x, y) \, d\mu(x) \, d\nu(y).$$

18.4. Absolute continuity and densities. The notation $\nu \ll \mu$ means

$$\mu(A) = 0 \implies \nu(A) = 0.$$

If μ, ν are sigma-finite and $\nu \ll \mu$, the Radon–Nikodym theorem gives a density

$$h = \frac{d\nu}{d\mu}$$

such that

$$\nu(A) = \int_A h \, d\mu, \quad \int f \, d\nu = \int fh \, d\mu.$$

19. WEAK DERIVATIVES

For smooth u and compactly supported smooth φ , integration by parts gives

$$\int_{\Omega} u \, \partial_i \varphi \, dx = - \int_{\Omega} (\partial_i u) \varphi \, dx.$$

The left side remains meaningful when u is not classically differentiable.

Definition 19.1. Let $u \in L^1_{\text{loc}}(\Omega)$. A function $v_i \in L^1_{\text{loc}}(\Omega)$ is the weak derivative $\partial_i u$ if

$$(19.1) \quad \int_{\Omega} u \, \partial_i \varphi \, dx = - \int_{\Omega} v_i \varphi \, dx$$

for every $\varphi \in C_c^\infty(\Omega)$.

Here $L^1_{\text{loc}}(\Omega)$ means integrability on every compact subset. The compact support of φ removes boundary terms.

Stopping point

Meeting 13 ended here. Meeting 14 begins by checking the definition for $u(x) = |x|$.

EXERCISES

Exercise 19.2. Why does (0.1) fail the hypotheses of dominated convergence?

Exercise 19.3. Give an example of a nonnegative measurable function to which Tonelli applies even though its integral is infinite.

Exercise 19.4. Why is the point mass δ_0 not absolutely continuous with respect to Lebesgue measure?

FURTHER READING

S. Axler, *Measure, Integration & Real Analysis*; H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*; L. C. Evans, *Partial Differential Equations*, Chapter 5.