

21-317 MEETING 12: THE OPERATIONAL INTEGRATION TOOLKIT I STUDENT NOTES AND EXERCISES

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ABSTRACT. Meeting 12 recalled the construction of the integral, introduced L^p and L^∞ , proved Hölder's inequality and probability-space norm monotonicity, and stated the monotone convergence, Fatou, and dominated convergence theorems. The class ended before the counterexample showing that pointwise convergence alone is insufficient.

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RECAP: CONSTRUCTION OF THE INTEGRAL

For a measurable set A , $\int \mathbf{1}_A \, d\mu = \mu(A)$. For a nonnegative simple function $s = \sum_j a_j \mathbf{1}_{A_j}$,

$$\int s \, d\mu = \sum_j a_j \mu(A_j).$$

For $f \geq 0$ measurable,

$$\int f \, d\mu = \sup \left\{ \int s \, d\mu : 0 \leq s \leq f, \, s \text{ simple} \right\}.$$

If $\int |f| \, d\mu < \infty$, then

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu.$$

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18. OPERATIONAL INTEGRATION TOOLKIT

18.1. L^p spaces. For $1 \leq p < \infty$,

$$(18.1) \quad \|f\|_p = \left(\int |f|^p d\mu \right)^{1/p}.$$

The space $L^p(\mu)$ identifies functions that agree almost everywhere. Also,

$$\|f\|_\infty = \inf\{M : |f| \leq M \text{ almost everywhere}\}.$$

Theorem 18.1 (Hölder). *If $p^{-1} + q^{-1} = 1$, then*

$$(18.2) \quad \int |fg| d\mu \leq \|f\|_p \|g\|_q.$$

Cauchy–Schwarz is the case $p = q = 2$. On a probability space, if $p \leq q$, then

$$(18.3) \quad \|f\|_p \leq \|f\|_q.$$

18.2. Convergence under the integral sign.

Theorem 18.2 (Monotone convergence). *If $0 \leq f_n \uparrow f$, then*

$$\int f_n d\mu \uparrow \int f d\mu.$$

Theorem 18.3 (Fatou). *If $f_n \geq 0$, then*

$$\int \liminf_n f_n d\mu \leq \liminf_n \int f_n d\mu.$$

Theorem 18.4 (Dominated convergence). *If $f_n \rightarrow f$ almost everywhere and $|f_n| \leq g$ for some $g \in L^1$, then*

$$\int f_n d\mu \rightarrow \int f d\mu.$$

Theorem	Hypothesis	Conclusion
Monotone convergence	$0 \leq f_n \uparrow f$	exact convergence, possibly to $+\infty$
Fatou	$f_n \geq 0$	one-sided lower bound
Dominated convergence	$f_n \rightarrow f$ a.e., $ f_n \leq g \in L^1$	convergence of finite integrals

EXERCISES

Exercise 18.5. Prove (18.3) from Hölder’s inequality.

Exercise 18.6. Which convergence theorem applies to x^n on $[0, 1]$?

FURTHER READING

S. Axler, *Measure, Integration & Real Analysis*, open access at <https://measure.axler.net/MIRA.pdf>.