

21-317 MEETING 11: MEASURE SPACES AND THE INTEGRAL REVISED STUDENT NOTES AND EXERCISES

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ABSTRACT. These notes record the material reached in Meeting 11: sigma-algebras, measures and countable additivity, measurable functions, null sets, and the construction of the integral from indicators and simple functions. Meeting 12 begins by recalling the integral and then develops L^p spaces and the main operational integration theorems.

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17. FROM FINITE SUMS TO MEASURE SPACES AND INTEGRALS

17.1. Sigma-algebras.

Definition 17.1. A sigma-algebra $\mathcal{A}$ on a set X is a collection of subsets containing $\emptyset$, closed under complements, and closed under countable unions. The pair $(X, \mathcal{A})$ is a measurable space.

It follows that $X \in \mathcal{A}$ and that $\mathcal{A}$ is closed under countable intersections. On a finite set one usually takes $\mathcal{A} = 2^X$. On $\mathbb{R}^d$, the Borel sigma-algebra is generated by the open sets.

17.2. Measures.

Definition 17.2. A measure is a map $\mu : \mathcal{A} \rightarrow [0, \infty]$ such that $\mu(\emptyset) = 0$ and, for pairwise disjoint $A_1, A_2, \dots$,

$$(17.1) \quad \mu \left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n).$$

A probability measure has $\mu(X) = 1$. A measure is sigma-finite if X is a countable union of finite-measure sets. Probability measures and Lebesgue measure on $\mathbb{R}^d$ are sigma-finite.

17.3. Measurable functions and almost everywhere. A function $f : X \rightarrow \mathbb{R}$ is measurable if inverse images of Borel sets are measurable. A set of measure zero is called null. A statement holds almost everywhere if it fails only on a null set.

17.4. Construction of the integral. For an indicator,

$$(17.2) \quad \int_X \mathbf{1}_A \, d\mu = \mu(A).$$

For a nonnegative simple function $s = \sum_{j=1}^m a_j \mathbf{1}_{A_j}$ with disjoint A_j ,

$$(17.3) \quad \int_X s \, d\mu = \sum_{j=1}^m a_j \mu(A_j).$$

For a nonnegative measurable function,

$$(17.4) \quad \int_X f \, d\mu = \sup \left\{ \int_X s \, d\mu : 0 \leq s \leq f, \, s \text{ simple} \right\}.$$

For a real-valued function, write $f = f^+ - f^-$, where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. If

$$(17.5) \quad \int_X |f| \, d\mu = \int_X f^+ \, d\mu + \int_X f^- \, d\mu < \infty,$$

then f is integrable and

$$(17.6) \quad \int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu.$$

On a finite probability space, this reduces to

$$\int_X f \, d\mu = \sum_{x \in X} f(x) \mu(x).$$

EXERCISES

Exercise 17.3. Show that a sigma-algebra is closed under countable intersections.

Exercise 17.4. Compute $\int (2\mathbf{1}_A + 5\mathbf{1}_B) \, d\mu$ when $A \cap B = \emptyset$.

Exercise 17.5. Why is the expression $+\infty - \infty$ excluded from the definition of a signed integral?

FURTHER READING

See S. Axler, *Measure, Integration & Real Analysis*, available at <https://measure.axler.net/MIRA.pdf>.