

21-317 MEETING 10: ENTROPY DISSIPATION AND FREE-ENERGY RELAXATION REVISED STUDENT NOTES REFLECTING THE MATERIAL COVERED

MATTHEW ROSENZWEIG

ABSTRACT. Meeting 10 resumed from the first derivative of entropy written at the end of Meeting 9. We completed the exact entropy-dissipation identity, compared the dissipation with square-root Dirichlet energy, derived exponential entropy and free-energy relaxation from the logarithmic Sobolev inequality, and contrasted the exact diffusion chain rule with the discrete inequality. The meeting ended at the close of Section 16. Measure spaces and integration begin in Meeting 11. Tensorization and Herbst concentration remain Round I presentation material.

CONTENTS

Recap from Meeting 9	1
16. Entropy decay and free-energy relaxation	2
16.1. Completing the dissipation identity	2
16.2. Comparison with square-root energy	2
Exercises	3
Sources and further reading	3
References	3

RECAP FROM MEETING 9

For a positive density h_0 relative to a reversible stationary law μ , set $h_t = P_t h_0$. Meeting 9 ended with

$$(0.1) \quad \frac{d}{dt} \text{Ent}_\mu(h_t) = \mathbb{E}_\mu[(1 + \log h_t) L h_t].$$

The logarithmic Sobolev inequality is

$$(0.2) \quad \text{Ent}_\mu(f^2) \leq \frac{2}{\rho} \mathcal{E}(f, f).$$

Date: September 16, 2026; revised September 18, 2026.

16. ENTROPY DECAY AND FREE-ENERGY RELAXATION

16.1. **Completing the dissipation identity.** Since μ is stationary,

$$(16.1) \quad \mathbb{E}_\mu[Lh_t] = 0.$$

Therefore equation (0.1) becomes

$$(16.2) \quad \frac{d}{dt} \text{Ent}_\mu(h_t) = -\mathcal{E}(h_t, \log h_t),$$

where detailed balance gives

$$(16.3) \quad \mathcal{E}(h, \log h) = \frac{1}{2} \sum_{x,y} \mu(x)q(x,y)(h(y) - h(x))(\log h(y) - \log h(x)) \geq 0.$$

Thus entropy is nonincreasing even before LSI supplies a quantitative rate.

16.2. **Comparison with square-root energy.**

Lemma 16.1. *For $a, b > 0$,*

$$(16.4) \quad (a - b)(\log a - \log b) \geq 4(\sqrt{a} - \sqrt{b})^2.$$

Applying this inequality edge by edge gives

$$(16.5) \quad \mathcal{E}(h, \log h) \geq 4\mathcal{E}(\sqrt{h}, \sqrt{h}).$$

Applying LSI to $f = \sqrt{h}$,

$$(16.6) \quad \text{Ent}_\mu(h) \leq \frac{1}{2\rho} \mathcal{E}(h, \log h).$$

Together with (16.2), this yields

$$(16.7) \quad \frac{d}{dt} \text{Ent}_\mu(h_t) \leq -2\rho \text{Ent}_\mu(h_t),$$

and therefore

$$(16.8) \quad \boxed{\text{Ent}_\mu(h_t) \leq e^{-2\rho t} \text{Ent}_\mu(h_0).}$$

If $\mu = \mu_\beta$ is a Gibbs law, then

$$(16.9) \quad D(\nu_t \| \mu_\beta) = \beta(\mathcal{F}_\beta(\nu_t) - \mathcal{F}_\beta(\mu_\beta)),$$

so entropy decay is exponential decay of excess free energy.

Discrete versus continuous

For a diffusion, the ordinary chain rule gives

$$(16.10) \quad \mathcal{E}(h, \log h) = 4\mathcal{E}(\sqrt{h}, \sqrt{h}).$$

For a discrete chain, equality generally fails and only inequality (16.5) is automatic.

Recorded endpoint

Meeting 10 ended here. Measure spaces and integration begin with Section 17 in Meeting 11.

EXERCISES

Exercise 16.2. Prove inequality (16.4) by setting $u = \sqrt{a/b}$ and reducing to $(u+1)\log u \geq 2(u-1)$.

Exercise 16.3. Derive (16.8) from (16.7) using an integrating factor.

SOURCES AND FURTHER READING

See Bakry–Gentil–Ledoux [1, Chs. 5–7] and Chafaï–Lehec [2, Chs. 1–3] for logarithmic Sobolev inequalities and entropy relaxation.

REFERENCES

- [1] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Springer, 2014.
- [2] D. Chafaï and J. Lehec, *Logarithmic Sobolev Inequalities Essentials*, revised 2023, <https://djalil.chafai.net/docs/M2/chafai-lehec-m2-lsie-lecture-notes.pdf>.