

21-317 MEETING 9: CYCLE, BOTTLENECKS, AND LOGARITHMIC SOBOLEV INEQUALITIES REVISED NOTES THROUGH THE START OF ENTROPY DISSIPATION

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ABSTRACT. Meeting 9 completed the cycle and bottleneck examples, introduced the logarithmic Sobolev inequality, and proved that its quadratic linearization is the Poincaré inequality. The class then introduced the density evolution $h_t = P_t h_0$ and began differentiating its entropy. It stopped before the exact entropy-dissipation identity was completed. Meeting 10 resumes that calculation; tensorization and Herbst concentration remain student-presentation material.

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RECAP FROM MEETING 8

For an irreducible finite reversible chain,

$$(0.1) \quad \lambda_1 = \inf_{f \text{ nonconstant}} \frac{\mathcal{E}(f, f)}{\text{Var}_\mu(f)},$$

and

$$(0.2) \quad \mathcal{E}(f, f) \geq \lambda \text{Var}_\mu(f) \iff \text{Var}_\mu(P_t f) \leq e^{-2\lambda t} \text{Var}_\mu(f).$$

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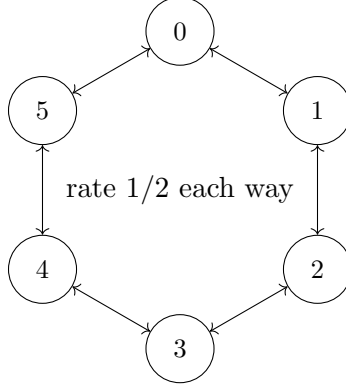
For the reset chain,

$$(0.3) \quad Lf = \mathbb{E}_\mu[f] - f, \quad P_t f = \mathbb{E}_\mu[f] + e^{-t}(f - \mathbb{E}_\mu[f]), \quad \lambda_1 = 1.$$

The next two examples show how the geometry of allowed transitions determines the size of the gap.

14. EXAMPLES AND GEOMETRIC INTERPRETATION, CONTINUED

14.3. **Cycle.** On $S = \mathbb{Z}/n\mathbb{Z}$, jump to each neighbor at rate $1/2$. The diagram shows $n = 6$.



The generator is

$$(14.1) \quad Lf(j) = \frac{1}{2}(f(j+1) + f(j-1) - 2f(j)).$$

For

$$(14.2) \quad \phi_k(j) = e^{2\pi i k j / n},$$

one computes

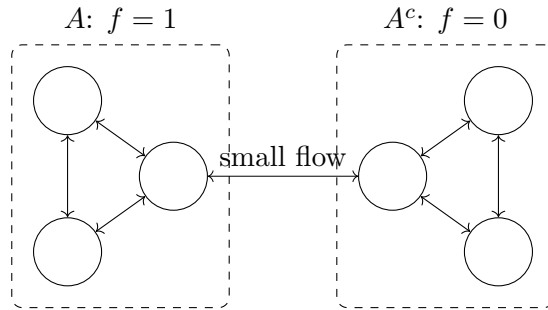
$$(14.3) \quad L\phi_k = -2 \sin^2\left(\frac{\pi k}{n}\right) \phi_k.$$

Hence

$$(14.4) \quad \lambda_1 = 2 \sin^2\left(\frac{\pi}{n}\right) \sim \frac{2\pi^2}{n^2}.$$

The relaxation time is therefore of order n^2 . This agrees with the diffusive rule of thumb: after time t , a symmetric random walk travels distance of order $\sqrt{t}$, so distance n requires time of order n^2 .

14.4. **Bottleneck test function.** Suppose two regions of substantial equilibrium mass communicate through only a small amount of equilibrium flow.



Define

$$(14.5) \quad Q(A, A^c) = \sum_{x \in A, y \notin A} \mu(x)q(x, y).$$

For $f = \mathbf{1}_A$,

$$(14.6) \quad \mathcal{E}(\mathbf{1}_A, \mathbf{1}_A) = Q(A, A^c),$$

and

$$(14.7) \quad \text{Var}_\mu(\mathbf{1}_A) = \mu(A)(1 - \mu(A)).$$

Therefore,

$$(14.8) \quad \lambda_1 \leq \frac{Q(A, A^c)}{\mu(A)(1 - \mu(A))}.$$

Large mass on both sides and small crossing flow force slow relaxation.

If

$$(14.9) \quad \Phi = \min_{0 < \mu(A) \leq 1/2} \frac{Q(A, A^c)}{\mu(A)},$$

then

$$(14.10) \quad \lambda_1 \leq 2\Phi.$$

15. LOGARITHMIC SOBOLEV INEQUALITY

15.1. Entropy and normalization. For $F \geq 0$, define

$$(15.1) \quad \text{Ent}_\mu(F) = \mathbb{E}_\mu[F \log F] - \mathbb{E}_\mu[F] \log \mathbb{E}_\mu[F].$$

If h is a density relative to μ , then

$$(15.2) \quad \text{Ent}_\mu(h) = D(h\mu \| \mu).$$

Definition 15.1 (Logarithmic Sobolev inequality). A reversible chain satisfies LSI with constant $\rho > 0$ if

$$(15.3) \quad \text{Ent}_\mu(f^2) \leq \frac{2}{\rho} \mathcal{E}(f, f)$$

for every observable f .

This convention makes the standard Gaussian constant equal to 1. Other sources use the reciprocal coefficient as the named constant.

15.2. Linearization gives Poincaré. Let g be centered and set $f_\varepsilon = 1 + \varepsilon g$. Taylor expansion gives

$$(15.4) \quad \text{Ent}_\mu(f_\varepsilon^2) = 2\varepsilon^2 \text{Var}_\mu(g) + O(\varepsilon^3),$$

while

$$(15.5) \quad \mathcal{E}(f_\varepsilon, f_\varepsilon) = \varepsilon^2 \mathcal{E}(g, g).$$

Substituting into (15.3) and letting $\varepsilon \rightarrow 0$,

$$(15.6) \quad \mathcal{E}(g, g) \geq \rho \text{Var}_\mu(g).$$

Thus LSI is an entropy-level strengthening of Poincaré, and $\rho \leq \lambda_1$.

16. BEGINNING ENTROPY DISSIPATION

Let $h_0 > 0$ be a density relative to μ . For a reversible chain, the density at time t is

$$(16.1) \quad h_t = P_t h_0.$$

Since $\mathbb{E}_\mu[h_t] = 1$,

$$(16.2) \quad \text{Ent}_\mu(h_t) = \mathbb{E}_\mu[h_t \log h_t].$$

Using $\partial_t h_t = Lh_t$, differentiation begins with

$$(16.3) \quad \frac{d}{dt} \text{Ent}_\mu(h_t) = \mathbb{E}_\mu[(1 + \log h_t)Lh_t].$$

Stopping point

Meeting 9 ended at equation (16.3). Meeting 10 first proves that the linear term vanishes and that

$$\frac{d}{dt} \text{Ent}_\mu(h_t) = -\mathcal{E}(h_t, \log h_t),$$

then obtains exponential entropy decay from the logarithmic Sobolev inequality.

EXERCISES

Exercise 16.1. Verify the cycle eigenvalue calculation and identify the first nonzero real eigenmodes.

Exercise 16.2. Use $f = \mathbf{1}_A$ in the Rayleigh quotient to derive the bottleneck upper bound for λ_1 .

Exercise 16.3. Verify the entropy linearization

$$\text{Ent}_\mu((1 + \varepsilon g)^2) = 2\varepsilon^2 \text{Var}_\mu(g) + O(\varepsilon^3)$$

for centered g .

SOURCES AND FURTHER READING

The cycle and bottleneck examples are standard; see Levin and Peres [3, Chs. 12–13]. The logarithmic Sobolev normalization, linearization, and entropy dissipation are treated by Bakry, Gentil, and Ledoux [1, Chs. 5–7] and Chafaï and Lehec [2, Chs. 1–3].

REFERENCES

- [1] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Springer, 2014.
- [2] D. Chafaï and J. Lehec, *Logarithmic Sobolev Inequalities Essentials*, revised 2023, <https://djalil.chafai.net/docs/M2/chafai-lehec-m2-lsie-lecture-notes.pdf>.
- [3] D. A. Levin and Y. Peres, with E. L. Wilmer, *Markov Chains and Mixing Times*, 2nd ed., AMS, 2017, <https://pages.uoregon.edu/dlevin/MARKOV/mcmt2e.pdf>.