

21-317 MEETING 8: THE POINCARÉ INEQUALITY IN ACTION SPECTRAL-GAP PROOFS AND ELEMENTARY MODEL CHAINS

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ABSTRACT. Meeting 7 ended after the statements of the Rayleigh-quotient characterization of the spectral gap and the equivalence between the Poincaré inequality and exponential variance relaxation. Meeting 8 recalled those statements, translated the functional-analysis terminology into finite-dimensional weighted linear algebra, defined irreducibility and reversibility, and proved both theorems. We then studied the symmetric two-state chain and the reset chain. The cycle and bottleneck examples were briefly advertised and are developed at the beginning of Meeting 9.

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RECAP FROM MEETING 7

For a finite stationary chain satisfying detailed balance with respect to μ ,

$$(0.1) \quad \mathcal{E}(f, g) := -\langle f, Lg \rangle_{L^2(\mu)},$$

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$$(0.2) \quad \mathcal{E}(f, f) = \frac{1}{2} \sum_{x, y \in S} \mu(x) q(x, y) (f(y) - f(x))^2,$$

and

$$(0.3) \quad \frac{d}{dt} \text{Var}_\mu(P_t f) = -2\mathcal{E}(P_t f, P_t f).$$

A Poincaré inequality is

$$(0.4) \quad \mathcal{E}(f, f) \geq \lambda \text{Var}_\mu(f).$$

Meeting 7 ended with the following statements.

Theorem 13.2 (Rayleigh quotient)

For an irreducible finite reversible chain,

$$(0.5) \quad \lambda_1 = \inf_{f \text{ nonconstant}} \frac{\mathcal{E}(f, f)}{\text{Var}_\mu(f)},$$

where λ_1 is the first positive eigenvalue of $-L$.

Theorem 13.3 (Poincaré inequality and relaxation)

For a fixed $\lambda > 0$, inequality (0.4) is equivalent to

$$(0.6) \quad \text{Var}_\mu(P_t f) \leq e^{-2\lambda t} \text{Var}_\mu(f), \quad t \geq 0.$$

13. TERMINOLOGY AND PROOFS OF THE SPECTRAL-GAP THEOREMS

13.1. Linear algebra and functional-analysis terminology. Let $S = \{x_1, \dots, x_m\}$, and let $\mu(x_i) > 0$. An observable $f : S \rightarrow \mathbb{R}$ is a vector in $\mathbb{R}^m$. The space $L^2(\mu)$ is this vector space equipped with

$$(13.1) \quad \langle f, g \rangle_{L^2(\mu)} = \sum_{i=1}^m \mu(x_i) f(x_i) g(x_i) = f^\top D_\mu g, \quad D_\mu := \text{diag}(\mu(x_1), \dots, \mu(x_m)).$$

Term	Meaning here	Linear-algebra translation
Hilbert space	$L^2(\mu)$ with its weighted inner product	$\mathbb{R}^m$ with a weighted dot product
operator	linear map on observables	matrix
self-adjoint	$\langle Af, g \rangle_\mu = \langle f, Ag \rangle_\mu$	symmetric in weighted coordinates
nonnegative	$\langle f, Af \rangle_\mu \geq 0$	positive semidefinite
kernel	$\{f : Af = 0\}$	nullspace
centered subspace	$f \perp 1$	$\mathbb{E}_\mu[f] = 0$
spectral theorem	orthonormal eigenbasis	diagonalization of a symmetric matrix
Dirichlet form	$\mathcal{E}(f, g) = \langle f, -Lg \rangle_\mu$	quadratic form of $-L$

Detailed balance is equivalent to

$$(13.2) \quad D_\mu L = L^\top D_\mu.$$

Consequently,

$$(13.3) \quad D_\mu^{1/2} L D_\mu^{-1/2}$$

is an ordinary symmetric matrix. Thus the spectral theorem below is the familiar symmetric-matrix theorem in weighted coordinates.

13.2. Irreducible and reversible.

Definition 13.1. The chain is *irreducible* if every state can be reached from every other state by a path of positive-rate transitions.

Under detailed balance with $\mu > 0$, a transition $x \rightarrow y$ has positive rate if and only if the reverse transition does. Hence irreducibility is connectedness of the undirected positive-rate graph.

If the graph is disconnected, a function may take different constants on different components. Such a function has zero energy but positive variance. Thus no positive Poincaré constant can hold across the full state space.

Definition 13.2. The chain is *reversible with respect to μ* if

$$(13.4) \quad \mu(x)q(x, y) = \mu(y)q(y, x).$$

This is the detailed-balance condition from Meetings 6–7. The word reversible comes from time reversal. In equilibrium,

$$(13.5) \quad \mu(x)P_t(x, y) = \mu(y)P_t(y, x),$$

so the two-time law is unchanged when the initial and final states are interchanged. Stationarity alone does not imply this; it may allow a circulating current.

13.3. Proof of Theorem 13.2. Irreducibility and the edge formula imply

$$(13.6) \quad \ker(-L) = \text{span}\{1\}.$$

The spectral theorem gives an orthonormal basis $(\phi_k)_{k=0}^{m-1}$ with

$$(13.7) \quad -L\phi_k = \lambda_k\phi_k, \quad 0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_{m-1}, \quad \phi_0 \equiv 1.$$

For $f^\circ = f - \mathbb{E}_\mu[f]$, write

$$(13.8) \quad f^\circ = \sum_{k=1}^{m-1} a_k \phi_k.$$

Then

$$(13.9) \quad \text{Var}_\mu(f) = \sum_{k=1}^{m-1} a_k^2$$

and

$$(13.10) \quad \begin{aligned} \mathcal{E}(f, f) &= \langle f^\circ, -Lf^\circ \rangle_{L^2(\mu)} \\ &= \sum_{k=1}^{m-1} \lambda_k a_k^2 \\ &\geq \lambda_1 \sum_{k=1}^{m-1} a_k^2 = \lambda_1 \text{Var}_\mu(f). \end{aligned}$$

Equality holds for any first eigenfunction, proving both sharpness and the Rayleigh-quotient formula.

Interpretation

The eigenfunction ϕ_k is a mode whose energy per unit variance is λ_k . The first nonconstant eigenspace is the least expensive nonconstant fluctuation and therefore the slowest mode.

13.4. Proof of Theorem 13.3. Assume the Poincaré inequality with constant λ , and set

$$(13.11) \quad Y(t) := \text{Var}_\mu(P_t f).$$

Variance dissipation gives

$$(13.12) \quad Y'(t) = -2\mathcal{E}(P_t f, P_t f) \leq -2\lambda Y(t).$$

Therefore,

$$(13.13) \quad \frac{d}{dt}(e^{2\lambda t} Y(t)) \leq 0,$$

which yields

$$(13.14) \quad \text{Var}_\mu(P_t f) \leq e^{-2\lambda t} \text{Var}_\mu(f).$$

Conversely, assume (0.6). Since equality holds at $t = 0$,

$$(13.15) \quad \frac{Y(t) - Y(0)}{t} \leq \frac{e^{-2\lambda t} - 1}{t} Y(0).$$

Letting $t \downarrow 0$ and using $Y'(0) = -2\mathcal{E}(f, f)$ gives

$$(13.16) \quad \mathcal{E}(f, f) \geq \lambda \text{Var}_\mu(f).$$

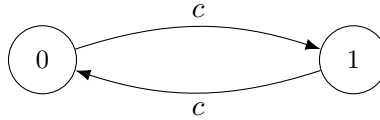
Moreover,

$$(13.17) \quad \text{Var}_\mu(P_t f) = \sum_{k=1}^{m-1} a_k^2 e^{-2\lambda_k t} \leq e^{-2\lambda_1 t} \text{Var}_\mu(f).$$

The first eigenspace realizes the sharp decay exponent.

14. EXAMPLES AND GEOMETRIC INTERPRETATION

14.1. Symmetric two-state chain. Let $S = \{0, 1\}$, with rate c in each direction and uniform stationary law.



The generator is

$$(14.1) \quad L = \begin{pmatrix} -c & c \\ c & -c \end{pmatrix}.$$

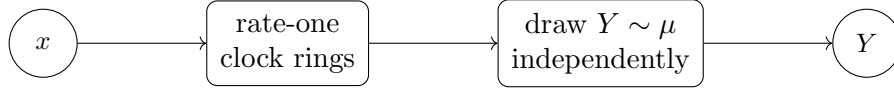
Its eigenvalues are 0 and $-2c$, so

$$(14.2) \quad \lambda_1 = 2c.$$

Equivalently,

$$(14.3) \quad \mathcal{E}(f, f) = \frac{c}{2}(f(1) - f(0))^2 = 2c \text{Var}_\mu(f).$$

14.2. Reset chain. At the rings of a rate-one clock, forget the current state and draw a fresh state from μ .



The generator is

$$(14.4) \quad Lf = \mathbb{E}_\mu[f] - f.$$

Every centered observable has eigenvalue -1 . Hence

$$(14.5) \quad \lambda_1 = 1,$$

and

$$(14.6) \quad P_t f = \mathbb{E}_\mu[f] + e^{-t}(f - \mathbb{E}_\mu[f]).$$

Thus

$$(14.7) \quad \text{Var}_\mu(P_t f) = e^{-2t} \text{Var}_\mu(f).$$

14.3. Preview of Meeting 9. Two further examples were advertised but not computed in Meeting 8.

- On a large cycle, local nearest-neighbor motion produces a spectral gap of order n^{-2} , so the relaxation time is of order n^2 .
- If two large regions communicate through only a small amount of equilibrium flow, an indicator-like observable has large variance and small energy, forcing a small spectral gap.

Meeting 9 begins with these examples and then introduces the logarithmic Sobolev inequality.

EXERCISES

Exercise 14.1 (Weighted symmetry). Show that detailed balance is equivalent to $D_\mu L = L^\top D_\mu$, and prove that $D_\mu^{1/2} L D_\mu^{-1/2}$ is symmetric.

Exercise 14.2 (Reducibility obstruction). Suppose the positive-rate graph has two connected components with positive μ -mass. Construct a nonconstant function with zero Dirichlet energy and explain why no positive Poincaré constant exists.

Exercise 14.3 (Mode-by-mode decay). Starting from the eigenbasis expansion, prove (13.17).

Exercise 14.4 (Reset chain). Show that the rates $q(x, y) = \mu(y)$ for $y \neq x$ produce (14.4) and satisfy detailed balance.

SOURCES AND FURTHER READING

The spectral-gap, two-state, and reset-chain material follows standard reversible Markov-chain theory; see Levin and Peres [3, Chs. 12–13]. The weighted Hilbert-space, self-adjointness, and semigroup language is developed by Bakry, Gentil, and Ledoux [1, Chs. 1 and 4] and Chafaï and Lehec [2, Ch. 1].

REFERENCES

- [1] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Springer, 2014.
- [2] D. Chafaï and J. Lehec, *Logarithmic Sobolev Inequalities Essentials*, revised 2023, <https://djalil.chafai.net/docs/M2/chafai-lehec-m2-lsie-lecture-notes.pdf>.
- [3] D. A. Levin and Y. Peres, with E. L. Wilmer, *Markov Chains and Mixing Times*, 2nd ed., AMS, 2017, <https://pages.uoregon.edu/dlevin/MARKOV/mcmt2e.pdf>.