

21-317 MEETING 7: DIRICHLET FORMS AND THE POINCARÉ INEQUALITY VARIANCE DISSIPATION, SPECTRAL GAP, AND RELAXATION

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ABSTRACT. Meeting 6 ended with a compressed introduction to stationarity, detailed balance, and self-adjointness for finite continuous-time Markov chains. We begin by recalling those notions carefully. We then define the reversible Dirichlet form, derive its edge representation, and prove the exact variance-dissipation identity. The Poincaré inequality closes this identity. In a finite irreducible reversible chain, its sharp constant is the first positive eigenvalue of $-L$, and it is equivalent to exponential variance relaxation. The final examples are included as continuation material.

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GENTLE RECAP THROUGH SECTION 11

For a finite state space S and transition rates $q(x, y)$, the generator is

$$(0.1) \quad Lf(x) = \sum_{y \neq x} q(x, y)(f(y) - f(x)),$$

and the semigroup is $P_t = e^{tL}$. Observables satisfy

$$(0.2) \quad \frac{d}{dt} P_t f = L P_t f,$$

while laws $\nu_t = \nu_0 P_t$ satisfy

$$(0.3) \quad \frac{d}{dt} \nu_t = \nu_t L.$$

These two evolutions are dual:

$$(0.4) \quad \sum_x \nu_t(x) f(x) = \sum_x \nu_0(x) P_t f(x).$$

A law μ is stationary precisely when

$$(0.5) \quad \mu P_t = \mu \text{ for all } t \geq 0 \iff \mu L = 0 \iff \mathbb{E}_\mu[Lf] = 0 \text{ for every } f.$$

Detailed balance is the stronger pairwise identity

$$(0.6) \quad \mu(x)q(x, y) = \mu(y)q(y, x), \quad x \neq y.$$

It says that every equilibrium flux $x \rightarrow y$ is cancelled by its reverse flux. Detailed balance implies stationarity and self-adjointness:

$$(0.7) \quad \langle f, Lg \rangle_{L^2(\mu)} = \langle Lf, g \rangle_{L^2(\mu)}.$$

The directed three-cycle is stationary but not reversible, so stationarity alone does not imply detailed balance.

12. DIRICHLET FORM AND VARIANCE DISSIPATION

12.1. Edge energy. Assume that μ satisfies detailed balance. Define

$$(12.1) \quad \mathcal{E}(f, g) := -\langle f, Lg \rangle_{L^2(\mu)}.$$

Proposition 12.1 (Edge representation). *For all observables f, g ,*

$$(12.2) \quad \mathcal{E}(f, g) = \frac{1}{2} \sum_{x, y} \mu(x)q(x, y)(f(y) - f(x))(g(y) - g(x)).$$

Consequently,

$$(12.3) \quad \mathcal{E}(f, f) = \frac{1}{2} \sum_{x, y} \mu(x)q(x, y)(f(y) - f(x))^2 \geq 0.$$

Proof. Starting from (12.1),

$$(12.4) \quad \mathcal{E}(f, g) = \sum_{x, y} \mu(x)q(x, y)f(x)(g(x) - g(y)).$$

Use detailed balance and exchange x, y to obtain

$$(12.5) \quad \mathcal{E}(f, g) = \sum_{x, y} \mu(x)q(x, y)f(y)(g(y) - g(x)).$$

Averaging gives (12.2); setting $g = f$ gives (12.3). $\square$

The energy is a weighted sum of squared changes across allowed transitions. If the positive-rate transition graph is connected, $\mathcal{E}(f, f) = 0$ if and only if f is constant.

12.2. Variance dissipation. Set $f_t = P_t f$. Stationarity gives

$$(12.6) \quad \mathbb{E}_\mu[f_t] = \mathbb{E}_\mu[f] =: m.$$

Theorem 12.2 (Variance dissipation). *For every observable f ,*

$$(12.7) \quad \frac{d}{dt} \text{Var}_\mu(P_t f) = -2\mathcal{E}(P_t f, P_t f).$$

Proof. Using the backward equation and preservation of the mean,

$$(12.8) \quad \begin{aligned} \frac{d}{dt} \text{Var}_\mu(f_t) &= \frac{d}{dt} \|f_t - m\|_{L^2(\mu)}^2 \\ &= 2 \langle f_t - m, Lf_t \rangle_{L^2(\mu)} \\ &= -2\mathcal{E}(f_t - m, f_t). \end{aligned}$$

Stationarity gives $\mathcal{E}(1, g) = -\mathbb{E}_\mu[Lg] = 0$, so the last expression is $-2\mathcal{E}(f_t, f_t)$. $\square$

The identity is exact but not closed: the left side contains variance and the right side contains energy. The Poincaré inequality compares them.

13. THE POINCARÉ INEQUALITY AND SPECTRAL GAP

13.1. Definition.

Definition 13.1 (Poincaré inequality). A reversible chain satisfies a Poincaré inequality with constant $\lambda > 0$ if

$$(13.1) \quad \mathcal{E}(f, f) \geq \lambda \text{Var}_\mu(f)$$

for every observable f .

Some sources write the reciprocal form

$$(13.2) \quad \text{Var}_\mu(f) \leq C_P \mathcal{E}(f, f), \quad C_P = \lambda^{-1}.$$

The convention must always be stated.

13.2. Variational characterization. Assume that the positive-rate transition graph is connected. Since L is self-adjoint and $\mathcal{E}(f, f) = \langle f, -Lf \rangle \geq 0$, the finite-dimensional spectral theorem gives an orthonormal eigenbasis satisfying

$$(13.3) \quad -L\phi_k = \lambda_k \phi_k, \quad 0 = \lambda_0 < \lambda_1 \leq \dots,$$

with $\phi_0 \equiv 1$. The number λ_1 is the spectral gap.

Theorem 13.2 (Rayleigh quotient).

$$(13.4) \quad \lambda_1 = \inf_{f \text{ nonconstant}} \frac{\mathcal{E}(f, f)}{\text{Var}_\mu(f)}.$$

Equivalently,

$$(13.5) \quad \mathcal{E}(f, f) \geq \lambda_1 \text{Var}_\mu(f)$$

for every f , and the constant is sharp.

Proof. Expand

$$(13.6) \quad f - \mathbb{E}_\mu[f] = \sum_{k \geq 1} a_k \phi_k.$$

Then

$$(13.7) \quad \text{Var}_\mu(f) = \sum_{k \geq 1} a_k^2,$$

and

$$(13.8) \quad \mathcal{E}(f, f) = \sum_{k \geq 1} \lambda_k a_k^2 \geq \lambda_1 \sum_{k \geq 1} a_k^2.$$

Equality holds for a first eigenfunction. □

13.3. Exponential relaxation.

Theorem 13.3. *For a fixed $\lambda > 0$, the Poincaré inequality (13.1) is equivalent to*

$$(13.9) \quad \text{Var}_\mu(P_t f) \leq e^{-2\lambda t} \text{Var}_\mu(f)$$

for every f and $t \geq 0$.

Proof. The Poincaré inequality and Theorem 12.2 give

$$(13.10) \quad \frac{d}{dt} \text{Var}_\mu(P_t f) \leq -2\lambda \text{Var}_\mu(P_t f).$$

Gronwall gives (13.9). Conversely, differentiate (13.9) at $t = 0$ and use (12.7). □

Thus, the spectral gap is simultaneously the first positive eigenvalue, the least energy per unit variance, and one half of the sharp exponential variance-decay rate.

14. EXAMPLES AND GEOMETRIC INTERPRETATION

The following examples are continuation material.

14.1. Two-state chain. If $S = \{0, 1\}$, $q(0, 1) = q(1, 0) = c$, and μ is uniform, then

$$(14.1) \quad \mathcal{E}(f, f) = \frac{c}{2} (f(1) - f(0))^2 = 2c \text{Var}_\mu(f).$$

Hence the gap is $2c$.

14.2. Reset chain. For

$$(14.2) \quad Lf = \mathbb{E}_\mu[f] - f,$$

every mean-zero function is an eigenfunction with eigenvalue -1 . Therefore,

$$(14.3) \quad P_t f = \mathbb{E}_\mu[f] + e^{-t}(f - \mathbb{E}_\mu[f]),$$

and the spectral gap is 1.

14.3. **Cycle.** For continuous-time simple random walk on $\mathbb{Z}/n\mathbb{Z}$,

$$(14.4) \quad Lf(j) = \frac{1}{2}(f(j+1) + f(j-1) - 2f(j)).$$

Fourier modes give

$$(14.5) \quad \lambda_1 = 2 \sin^2\left(\frac{\pi}{n}\right) \sim \frac{2\pi^2}{n^2}.$$

The relaxation time therefore grows on the diffusive scale n^2 .

14.4. **Bottleneck test.** For $A \subset S$, set

$$(14.6) \quad Q(A, A^c) = \sum_{x \in A, y \notin A} \mu(x)q(x, y).$$

Testing the Rayleigh quotient with $\mathbf{1}_A$ gives

$$(14.7) \quad \lambda_1 \leq \frac{Q(A, A^c)}{\mu(A)(1 - \mu(A))}.$$

A set with large mass and little outward flow forces slow relaxation.

WHAT TO RETAIN

- (1) Detailed balance turns the generator into a self-adjoint operator and produces the nonnegative edge energy $\mathcal{E}$.
- (2) Variance satisfies the exact identity $\frac{d}{dt} \text{Var} = -2\mathcal{E}$.
- (3) The Poincaré inequality compares global variance with local edge energy.
- (4) For an irreducible finite reversible chain, the sharp Poincaré constant is the first positive eigenvalue of $-L$.
- (5) Poincaré coercivity is equivalent to exponential variance relaxation.
- (6) Small spectral gaps arise from slowly varying modes or geometric bottlenecks.

EXERCISES

Exercise 14.1. Derive the edge representation (12.2) from detailed balance.

Exercise 14.2. Characterize all zero-energy functions when the positive-rate graph has more than one connected component.

Exercise 14.3. Reproduce the proof of the variance-dissipation identity and identify where stationarity and the backward equation enter.

Exercise 14.4. Compute the Poincaré constant of the symmetric two-state chain.

Exercise 14.5. Derive the reset-chain semigroup (14.3) and verify its exact variance decay.

Exercise 14.6. Verify the cycle eigenvalue calculation and deduce (14.5).

Exercise 14.7. Prove the bottleneck upper bound (14.7) by testing with an indicator function.

SOURCES AND FURTHER READING

For reversible finite chains, spectral gaps, Rayleigh quotients, and relaxation, see Levin and Peres [3, Chs. 12–13]. For the semigroup and Dirichlet-form viewpoint, see Bakry, Gentil, and Ledoux [1, Chs. 1 and 4] and Chafaï and Lehec [2, Ch. 1]. Norris [4] is a standard reference for finite continuous-time Markov chains.

REFERENCES

- [1] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Springer, 2014.
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- [3] D. A. Levin and Y. Peres, with E. L. Wilmer, *Markov Chains and Mixing Times*, 2nd ed., AMS, 2017, <https://pages.uoregon.edu/dlevin/MARKOV/mcmt2e.pdf>.
- [4] J. R. Norris, *Markov Chains*, Cambridge University Press, 1997.