

21-317 MEETING 6: CONTINUOUS-TIME MARKOV CHAINS II GENERATORS, JUMP RATES, STATIONARITY, AND REVERSIBILITY

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ABSTRACT. Meeting 5 ended after constructing a continuous-time process by running a discrete Markov chain on a Poisson clock. Meeting 6 derived the generator, distinguished attempted updates from actual jumps and exit rates, extended the construction to arbitrary finite rate systems by uniformization, and introduced the forward equation, stationary laws, and detailed balance. The meeting ended after a compressed treatment of Section 11. Meeting 7 begins by recalling that material before introducing Dirichlet forms and variance dissipation.

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RECAP THROUGH SECTION 8

Let $(Y_n)_{n \geq 0}$ have transition kernel K , and let $(N_t)_{t \geq 0}$ be an independent Poisson process of rate a . Meeting 5 defined

$$(0.1) \quad X_t = Y_{N_t}.$$

and proved

$$(0.2) \quad P_t f(x) := \mathbb{E}_x[f(X_t)] = e^{-at} \sum_{n=0}^{\infty} \frac{(at)^n}{n!} K^n f(x) = e^{at(K-I)} f(x).$$

Thus

$$(0.3) \quad P_0 = I, \quad P_{s+t} = P_s P_t.$$

If $\nu_t = \text{Law}(X_t)$, then $\nu_t = \nu_0 P_t$, and if $\mu K = \mu$, then $\mu P_t = \mu$ for all $t \geq 0$.

9. THE INFINITESIMAL GENERATOR AND ACTUAL EXIT RATES

9.1. Short-time expansion. For $h \downarrow 0$,

$$(9.1) \quad \mathbb{P}(N_h = 0) = 1 - ah + O(h^2),$$

$$(9.2) \quad \mathbb{P}(N_h = 1) = ah + O(h^2),$$

$$(9.3) \quad \mathbb{P}(N_h \geq 2) = O(h^2).$$

Using the Poisson-mixture formula,

$$(9.4) \quad P_h f = f + ah(K - I)f + O(h^2) \|f\|_{\infty}.$$

Definition 9.1 (Infinitesimal generator). The generator of (P_t) is

$$(9.5) \quad Lf := \lim_{h \downarrow 0} \frac{P_h f - f}{h}.$$

Equation (9.4) gives

$$(9.6) \quad L = a(K - I).$$

Since

$$(9.7) \quad Kf(x) - f(x) = \mathbb{E}_x[f(Y_1) - f(Y_0)],$$

we can interpret

$$(9.8) \quad \boxed{Lf(x) = \text{update rate} \times \text{expected change of } f \text{ per update from } x.}$$

The semigroup satisfies

$$(9.9) \quad \frac{d}{dt} P_t f = L P_t f = P_t L f.$$

9.2. Attempted updates and actual jumps. A rate- a clock ring is an update opportunity. If $K(x, x) > 0$, an update may leave the state unchanged. For $y \neq x$, define the actual transition rate

$$(9.10) \quad q(x, y) := aK(x, y).$$

The total exit rate from x is

$$(9.11) \quad r(x) := \sum_{y \neq x} q(x, y) = a(1 - K(x, x)).$$

Thus the events counted by $r(x)$ are actual departures from x .

Proposition 9.2 (Holding time and destination). *If T_x is the time until the process first leaves x , then*

$$(9.12) \quad \mathbb{P}_x(T_x > t) = e^{-r(x)t}.$$

Hence $T_x \sim \text{Exp}(r(x))$ and $\mathbb{E}_x[T_x] = 1/r(x)$ when $r(x) > 0$. Conditional on leaving,

$$(9.13) \quad \mathbb{P}_x(\text{next state is } y \mid \text{leave } x) = \frac{q(x, y)}{r(x)}.$$

Proof. Conditional on $N_t = n$, the chain has not left x precisely when all n updates are self-transitions. Therefore,

$$(9.14) \quad \mathbb{P}_x(T_x > t) = \sum_{n=0}^{\infty} e^{-at} \frac{(at)^n}{n!} K(x, x)^n = e^{-at(1-K(x, x))}.$$

This is (9.12). The conditional destination probability is $K(x, y)/(1 - K(x, x)) = q(x, y)/r(x)$. $\square$

The chain at x is therefore described by

$$(9.15) \quad \boxed{\begin{array}{l} \text{holding time at } x \sim \text{Exp}(r(x)), \\ \text{next destination} \sim \left(\frac{q(x, y)}{r(x)} \right)_{y \neq x}. \end{array}}$$

10. GENERAL FINITE CONTINUOUS-TIME MARKOV CHAINS

10.1. Arbitrary rates. Let S be finite, and assign a rate

$$(10.1) \quad q(x, y) \geq 0$$

for every distinct $x, y \in S$. Define

$$(10.2) \quad r(x) := \sum_{y \neq x} q(x, y)$$

and

$$(10.3) \quad Lf(x) := \sum_{y \neq x} q(x, y)(f(y) - f(x)).$$

As a matrix,

$$(10.4) \quad L(x, y) = q(x, y) \quad (y \neq x), \quad L(x, x) = -r(x).$$

Every row of L sums to zero.

10.2. Uniformization.

Theorem 10.1 (Uniformization). *Let*

$$(10.5) \quad R := \max_x r(x),$$

and choose $a \geq R$. Define

$$(10.6) \quad K(x, y) = \begin{cases} q(x, y)/a, & y \neq x, \\ 1 - r(x)/a, & y = x. \end{cases}$$

Then K is a stochastic matrix and

$$(10.7) \quad L = a(K - I).$$

Consequently,

$$(10.8) \quad P_t = e^{tL} = e^{-at} \sum_{n=0}^{\infty} \frac{(at)^n}{n!} K^n.$$

Proof. The entries in (10.6) are nonnegative and each row sums to one. For $y \neq x$, the (x, y) entry of $a(K - I)$ is $q(x, y)$. Its diagonal entry is $-r(x)$. $\square$

Uniformization uses one master clock of rate a . At a state with smaller exit rate, the unused part of the master rate becomes a self-transition.

10.3. Competing clocks. Equivalently, while the process is at x , give each possible transition $x \rightarrow y$ an independent clock

$$(10.9) \quad T_{x,y} \sim \text{Exp}(q(x, y)).$$

The first clock to ring determines the jump. Independence gives

$$(10.10) \quad \min_{y \neq x} T_{x,y} \sim \text{Exp} \left(\sum_{y \neq x} q(x, y) \right) = \text{Exp}(r(x)),$$

and the probability that the winning clock is the one for $x \rightarrow y$ is $q(x, y)/r(x)$.

10.4. Small-time transition probabilities. Uniformization gives, for $y \neq x$,

$$(10.11) \quad \mathbb{P}_x(X_h = y) = q(x, y)h + O(h^2),$$

and

$$(10.12) \quad \mathbb{P}_x(X_h = x) = 1 - r(x)h + O(h^2).$$

The $O(h^2)$ contribution comes from two or more master-clock rings. Hence

$$(10.13) \quad \mathbb{E}_x[f(X_h)] = f(x) + h \sum_{y \neq x} q(x, y)(f(y) - f(x)) + O(h^2) \|f\|_{\infty},$$

which recovers (10.3) from the derivative definition.

11. STATIONARITY AND REVERSIBILITY IN CONTINUOUS TIME

11.1. **Observables and laws.** For a law ν and observable f , set

$$(11.1) \quad \langle \nu, f \rangle := \sum_x \nu(x) f(x).$$

If $\nu_t = \nu_0 P_t$, then

$$(11.2) \quad \langle \nu_t, f \rangle = \langle \nu_0, P_t f \rangle.$$

The action of L on laws is

$$(11.3) \quad (\nu L)(y) := \sum_x \nu(x) L(x, y).$$

Then

$$(11.4) \quad \frac{d}{dt} \nu_t = \nu_t L,$$

while observables satisfy the backward equation (9.9).

Proposition 11.1 (Stationarity criterion). *A probability law μ is stationary if and only if*

$$(11.5) \quad \mu L = 0.$$

Equivalently,

$$(11.6) \quad \mathbb{E}_\mu[Lf] = 0$$

for every observable f .

Proof. If $\mu P_t = \mu$, differentiate at $t = 0$. Conversely, if $\mu L = 0$, then $\mu L^n = 0$ for $n \geq 1$, so $\mu e^{tL} = \mu$. $\square$

11.2. Detailed balance.

Definition 11.2. A law μ satisfies detailed balance if

$$(11.7) \quad \mu(x) q(x, y) = \mu(y) q(y, x)$$

for all $x \neq y$.

Proposition 11.3. *Detailed balance implies stationarity and makes L self-adjoint in $L^2(\mu)$:*

$$(11.8) \quad \langle f, Lg \rangle_{L^2(\mu)} = \langle Lf, g \rangle_{L^2(\mu)}.$$

Proof. Use (11.7) to interchange x and y in the double sum defining $\langle f, Lg \rangle$. Taking $f = 1$ then gives $\mathbb{E}_\mu[Lg] = 0$, hence stationarity. $\square$

Stationarity does not imply detailed balance; the directed three-cycle from Meeting 5 is the standard counterexample.

MEETING ENDPOINT

Meeting 6 ended with the statement that detailed balance implies stationarity and self-adjointness of L in $L^2(\mu)$. The next meeting begins with a gentle review of the forward equation, stationarity, detailed balance, and the distinction between a stationary circulation and a reversible equilibrium.

WHAT TO RETAIN

- (1) The generator is the instantaneous derivative of the semigroup and equals $a(K - I)$ under Poissonization.
- (2) The rate $q(x, y)$ counts actual jumps $x \rightarrow y$; the exit rate $r(x)$ counts all departures from x .
- (3) Uniformization constructs every finite continuous-time chain from one sufficiently fast Poisson clock.
- (4) The backward equation evolves observables, while the forward equation evolves laws.
- (5) A law is stationary if and only if $\mu L = 0$.
- (6) Detailed balance is stronger than stationarity and makes the generator self-adjoint.

EXERCISES

Exercise 11.4. Derive $L = a(K - I)$ from the short-time Poisson probabilities.

Exercise 11.5. Suppose a rate-10 master clock applies a kernel satisfying $K(x, x) = 0.7$. Compute the actual exit rate and mean holding time at x .

Exercise 11.6. Verify the uniformization theorem for a system of rates $q(x, y)$.

Exercise 11.7. Use uniformization to prove the small-time formulas (10.11)–(10.12).

Exercise 11.8. Prove that $\mu L = 0$ is equivalent to $\mu P_t = \mu$ for every $t \geq 0$.

Exercise 11.9. Prove directly that detailed balance implies stationarity and self-adjointness of L in $L^2(\mu)$.

SOURCES AND FURTHER READING

For finite-state generators, uniformization, holding times, and the forward and backward equations, see Norris [4] and Levin and Peres [3]. For reversible semigroups and the transition from detailed balance to Dirichlet forms, see Bakry, Gentil, and Ledoux [1] and Chafaï and Lehec [2].

REFERENCES

- [1] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Springer, 2014.
- [2] D. Chafaï and J. Lehec, *Logarithmic Sobolev Inequalities Essentials*, revised 2023, <https://djalil.chafai.net/docs/M2/chafai-lehec-m2-lsie-lecture-notes.pdf>.
- [3] D. A. Levin and Y. Peres, with E. L. Wilmer, *Markov Chains and Mixing Times*, 2nd ed., AMS, 2017, <https://pages.uoregon.edu/dlevin/MARKOV/mcmt2e.pdf>.
- [4] J. R. Norris, *Markov Chains*, Cambridge University Press, 1997.