

21-317 MEETING 1: WHY SHOULD A FUNCTION BE ALMOST CONSTANT? STUDENT NOTES AND EXERCISES

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ABSTRACT. Meeting 1 gave a panorama of the course rather than beginning the formal theory. A high-dimensional Gaussian shell illustrated concentration of measure. The uniform two-point space gave an exact local-to-global identity. We then saw the two principal templates that will recur throughout the course: Poincaré inequalities combine with variance dissipation to give exponential relaxation, while logarithmic Sobolev inequalities control exponential moments and hence concentration. Relative entropy and Gibbs free energy provided the physical interpretation.

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1. THE GUIDING QUESTION

The central question of the course is

(1.1) Why should a function be almost constant?

A functional inequality provides a quantitative form of the principle

(1.2) $\text{global fluctuation} \leq \text{constant} \times \text{local energy}.$

The meaning of “global fluctuation” and “local energy” changes with the setting:

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	Continuous setting	Discrete setting
Local variation	derivative or gradient	finite difference across an edge
Energy	integral of squared gradient	weighted sum of squared differences
Global quantity	variance, entropy, or an L^p norm	variance, entropy, or boundary size

The first meeting displayed the architecture through examples. Precise definitions and full proofs of the stated inputs appear later.

2. A GAUSSIAN VECTOR IN ONE MILLION DIMENSIONS

Let $G = (G_1, \dots, G_n)$ have independent standard Gaussian coordinates. Its law is

$$(2.1) \quad \gamma_n(\mathrm{d}x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2} \mathrm{d}x.$$

Theorem 2.1 (Gaussian concentration, preview). *If $F : \mathbb{R}^n \rightarrow \mathbb{R}$ is 1-Lipschitz, then*

$$(2.2) \quad \mathbb{P}(|F(G) - \mathbb{E}[F(G)]| \geq r) \leq 2e^{-r^2/2}$$

for every $r \geq 0$. The bound is independent of n .

The Euclidean norm is 1-Lipschitz because

$$(2.3) \quad ||x| - |y|| \leq |x - y|.$$

Thus, for $n = 10^6$,

$$(2.4) \quad \mathbb{P}(|G| - \mathbb{E}[|G|] \geq 5) \leq 2e^{-25/2} < 8 \times 10^{-6}.$$

Since $\mathbb{E}[|G|] \sim \sqrt{n}$, the typical radius is close to 1000. Almost all Gaussian mass is therefore concentrated in a shell whose thickness is negligible compared with its radius.

Why this is surprising

The ambient dimension is one million, but the deviation scale in (2.4) is dimension-free. The course will explain how a local regularity condition on F produces such a global probabilistic statement.

3. THE TWO-POINT MODEL

Let

$$(3.1) \quad \Omega = \{-1, +1\}, \quad \mu(-1) = \mu(+1) = \frac{1}{2}.$$

If $f(-1) = a$ and $f(+1) = b$, then

$$(3.2) \quad \mathbb{E}_\mu[f] = \frac{a+b}{2}.$$

Proposition 3.1 (Exact two-point identity). *For every $f : \Omega \rightarrow \mathbb{R}$,*

$$(3.3) \quad \mathrm{Var}_\mu(f) = \frac{1}{4}(a-b)^2.$$

If

$$(3.4) \quad \nabla_{\mathrm{d}f} := \frac{f(+1) - f(-1)}{2},$$

then

$$(3.5) \quad \text{Var}_\mu(f) = |\nabla_{\text{d}} f|^2.$$

Proof. Using (3.2),

$$\begin{aligned} \text{Var}_\mu(f) &= \frac{1}{2} \left(a - \frac{a+b}{2} \right)^2 + \frac{1}{2} \left(b - \frac{a+b}{2} \right)^2 \\ &= \frac{1}{4} (a-b)^2. \end{aligned}$$

Equation (3.5) follows from (3.4). $\square$

On this smallest connected space, global fluctuation is exactly local squared variation. A Poincaré inequality generalizes this identity.

4. POINCARÉ INEQUALITIES AND RELAXATION

Suppose $(P_t)_{t \geq 0}$ is a reversible Markov evolution with invariant probability measure μ and Dirichlet form $\mathcal{E}$. Later meetings develop these notions rigorously. The panorama used the exact identity

$$(4.1) \quad \frac{d}{dt} \text{Var}_\mu(P_t f) = -2\mathcal{E}(P_t f, P_t f).$$

A Poincaré inequality with constant $\lambda > 0$ is

$$(4.2) \quad \text{Var}_\mu(g) \leq \frac{1}{\lambda} \mathcal{E}(g, g).$$

Proposition 4.1 (Relaxation consequence). *If (4.1) and (4.2) hold, then*

$$(4.3) \quad \text{Var}_\mu(P_t f) \leq e^{-2\lambda t} \text{Var}_\mu(f).$$

Proof. Combining the two inputs gives

$$(4.4) \quad \frac{d}{dt} \text{Var}_\mu(P_t f) \leq -2\lambda \text{Var}_\mu(P_t f).$$

Therefore $e^{2\lambda t} \text{Var}_\mu(P_t f)$ is nonincreasing, which proves (4.3). $\square$

Interpretation of λ

In a finite reversible chain, λ is the first nonzero eigenvalue of $-L$. It is also the coercivity constant in the Poincaré inequality and the inverse time scale for variance relaxation.

5. ENTROPY AND GIBBS FREE ENERGY

Let Ω be finite and let $E : \Omega \rightarrow \mathbb{R}$ be an energy. For inverse temperature $\beta > 0$, define

$$(5.1) \quad Z_\beta := \sum_x e^{-\beta E(x)}, \quad \mu_\beta(x) := \frac{e^{-\beta E(x)}}{Z_\beta}.$$

For a probability law ν , define

$$(5.2) \quad H(\nu) := - \sum_x \nu(x) \log \nu(x)$$

and

$$(5.3) \quad \mathcal{F}_\beta(\nu) := \sum_x E(x)\nu(x) - \frac{1}{\beta} H(\nu).$$

The relative entropy of ν with respect to a positive law μ is

$$(5.4) \quad D(\nu\|\mu) := \sum_x \nu(x) \log \frac{\nu(x)}{\mu(x)}.$$

Proposition 5.1 (Free-energy identity, preview). *For every probability law ν ,*

$$(5.5) \quad D(\nu\|\mu_\beta) = \beta(\mathcal{F}_\beta(\nu) - \mathcal{F}_\beta(\mu_\beta)),$$

where

$$(5.6) \quad \mathcal{F}_\beta(\mu_\beta) = -\frac{1}{\beta} \log Z_\beta.$$

Proof. From (5.1),

$$(5.7) \quad \log \mu_\beta(x) = -\beta E(x) - \log Z_\beta.$$

Hence

$$\begin{aligned} D(\nu\|\mu_\beta) &= \sum_x \nu(x) \log \nu(x) + \beta \sum_x E(x)\nu(x) + \log Z_\beta \\ &= \beta \mathcal{F}_\beta(\nu) + \log Z_\beta. \end{aligned}$$

Taking $\nu = \mu_\beta$ proves (5.6); substitution gives (5.5). $\square$

Physical interpretation

Expected energy favors low-energy states, while $-\beta^{-1}H(\nu)$ favors laws spread over many states. Relative entropy to Gibbs equilibrium measures excess free energy. The nonnegativity of relative entropy, proved later using Jensen's inequality, identifies μ_β as the free-energy minimizer.

6. LOGARITHMIC SOBOLEV INEQUALITY AND THE HERBST ARGUMENT

For a nonnegative integrable function h , define

$$(6.1) \quad \text{Ent}_{\gamma_n}(h) := \int h \log h \, d\gamma_n - \left(\int h \, d\gamma_n \right) \log \left(\int h \, d\gamma_n \right).$$

Theorem 6.1 (Gaussian logarithmic Sobolev inequality, stated input). *For every sufficiently regular $g : \mathbb{R}^n \rightarrow \mathbb{R}$,*

$$(6.2) \quad \text{Ent}_{\gamma_n}(e^g) \leq \frac{1}{2} \int e^g |\nabla g|^2 \, d\gamma_n.$$

The essential feature is that the constant $1/2$ does not depend on dimension.

Theorem 6.2 (Herbst argument). *The inequality (6.2) implies Theorem 2.1.*

Proof. Let F be smooth with $|\nabla F| \leq 1$ and define

$$(6.3) \quad \psi(\theta) := \log \int e^{\theta F} d\gamma_n.$$

Apply (6.2) with $g = \theta F$. After division by $\int e^{\theta F} d\gamma_n$,

$$(6.4) \quad \theta \psi'(\theta) - \psi(\theta) \leq \frac{\theta^2}{2}.$$

Thus

$$(6.5) \quad \frac{d}{d\theta} \left(\frac{\psi(\theta)}{\theta} \right) \leq \frac{1}{2}.$$

Since $\psi(\theta)/\theta \rightarrow \mathbb{E}[F(G)]$ as $\theta \downarrow 0$,

$$(6.6) \quad \log \mathbb{E} \left[e^{\theta(F(G) - \mathbb{E}[F(G)])} \right] \leq \frac{\theta^2}{2}.$$

For $r > 0$, exponential Markov inequality yields

$$(6.7) \quad \mathbb{P}(F(G) - \mathbb{E}[F(G)] \geq r) \leq \exp \left(-\theta r + \frac{\theta^2}{2} \right).$$

The optimal choice is $\theta = r$, giving

$$(6.8) \quad \mathbb{P}(F(G) - \mathbb{E}[F(G)] \geq r) \leq e^{-r^2/2}.$$

Apply the argument to $-F$ for the two-sided bound. $\square$

The local-to-global conversion

The hypothesis $|\nabla F| \leq 1$ is a deterministic local condition. The conclusion controls global deviations of the random variable $F(G)$. The logarithmic Sobolev inequality is the bridge between them.

7. COURSE MAP

The continuous and discrete theories share a common variational structure.

	Continuous models	Discrete models
Spaces	Euclidean domains, Gaussian space, manifolds, Gibbs measures	graphs, the discrete cube, finite chains, spin systems
Local energy	gradient or derivative energy	finite-difference or edge energy
Dynamics	heat, Ornstein–Uhlenbeck, Langevin, Fokker–Planck	random walk, Glauber dynamics, sampling chains
Geometry	perimeter and isoperimetry	edge boundary, conductance, Cheeger constants

The recurring chain is

$$(7.1) \quad \boxed{\text{geometry, convexity, or independence}} \implies \boxed{\text{functional inequality}} \\ \implies \boxed{\text{concentration, smoothing, or relaxation}}.$$

For every theorem, ask:

- (1) What global quantity is being controlled?

- (2) What local energy or geometric quantity controls it?
- (3) What conclusion follows from the constant?

8. EXERCISES

Exercise 8.1 (Two-point identity). Prove Proposition 3.1 directly and determine when the variance vanishes.

Exercise 8.2 (Relaxation inequality). If $Y(t) \geq 0$ and $Y'(t) \leq -2\lambda Y(t)$, prove $Y(t) \leq e^{-2\lambda t} Y(0)$.

Exercise 8.3 (Free-energy algebra). Derive (5.5) from (5.1). Identify the step using $\sum_x \nu(x) = 1$.

Exercise 8.4 (Chernoff optimization). Minimize $-\theta r + \theta^2/2$ over $\theta > 0$ and recover the exponent in (6.8).

Exercise 8.5 (Euclidean norm). Prove (2.3). Explain how smooth approximation can remove the nondifferentiability of $|x|$ at the origin in the Herbst argument.

9. SOURCES AND FURTHER READING

The semigroup viewpoint and the relaxation consequences of functional inequalities are developed systematically by Bakry, Gentil, and Ledoux [1] and Chafaï and Lehec [3]. Gross proved the dimension-free Gaussian logarithmic Sobolev inequality [4]. Ledoux gives a detailed account of the Herbst argument and concentration [5]. Chafaï’s mini-course connects entropy, Gibbs measures, free energy, logarithmic Sobolev inequalities, and concentration [2].

REFERENCES

- [1] D. Bakry, I. Gentil, and M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Springer, 2014.
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- [4] L. Gross, Logarithmic Sobolev inequalities, *Amer. J. Math.* **97** (1975), 1061–1083.
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