

A SPECTRAL GAP FOR THE UNLABELED INFINITE GINIBRE INTERACTING BROWNIAN MOTION

ABSTRACT. Let μ_{Gin} be the infinite Ginibre point process and let $\mathcal{E}^{\mu_{\text{Gin}}}$ be the canonical strongly local Dirichlet form of the unlabeled Ginibre interacting Brownian motion. We prove

$$\text{Var}_{\mu_{\text{Gin}}}(F) \leq \mathcal{E}^{\mu_{\text{Gin}}}(F, F)$$

for every function in the closed form domain, under the convention

$$\mathcal{E}^{\mu_{\text{Gin}}}(F, F) = \frac{1}{2} \int_{\text{Conf}(\mathbb{C})} \sum_{x \in \gamma} |\nabla_x F(\gamma)|^2 \mu_{\text{Gin}}(d\gamma).$$

Thus the unlabeled infinite Ginibre diffusion has spectral gap at least one. The proof first establishes an exact, particle-number-uniform Poincaré inequality on the permutation-invariant sector of the finite complex Ginibre eigenvalue law. Multiplication by the Vandermonde determinant maps this sector isometrically into the alternating sector of Gaussian L^2 . The Gaussian $\bar{\partial}$ spectral gap, combined with a common-phase decomposition and the minimal holomorphic degree of an alternating polynomial, gives the finite inequality with sharp constant. Local determinantal convergence and uniform exponential moments of compact-set counts then pass the estimate to the infinite point field.

1. STATEMENT AND NORMALIZATION

Let

$$(1) \quad \mu_N(dZ) = \frac{1}{Z_N} e^{-\sum_{i=1}^N |z_i|^2} \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 \prod_{i=1}^N dA(z_i), \quad Z = (z_1, \dots, z_N) \in \mathbb{C}^N,$$

where dA is planar Lebesgue measure. This is the unscaled complex Ginibre eigenvalue law, whose bulk density near the origin is of order one.

For a real permutation-invariant function f put

$$(2) \quad \mathcal{E}_N(f, f) := \frac{1}{2} \int_{\mathbb{C}^N} \sum_{i=1}^N |\nabla_i f|^2 d\mu_N.$$

Theorem 1.1 (Exact finite symmetric-sector gap). *For every $N \geq 1$ and every real permutation-invariant f in the closed form domain,*

$$(3) \quad \text{Var}_{\mu_N}(f) \leq \mathcal{E}_N(f, f).$$

The constant is sharp: the symmetric-sector spectral-gap rate is exactly one.

Let μ_{Gin} be the translation-invariant determinantal point process with kernel

$$(4) \quad K(z, w) = \frac{1}{\pi} \exp\left(z\bar{w} - \frac{|z|^2 + |w|^2}{2}\right)$$

with respect to planar Lebesgue measure. On the standard smooth local cylinder core, define

$$(5) \quad \mathcal{E}^{\mu_{\text{Gin}}}(F, F) := \frac{1}{2} \int_{\text{Conf}(\mathbb{C})} \sum_{x \in \gamma} |\nabla_x F(\gamma)|^2 \mu_{\text{Gin}}(d\gamma).$$

The pre-form is closable and its closure is the Dirichlet form of the unlabeled Ginibre interacting Brownian motion; see [2, 3].

Theorem 1.2 (Infinite Ginibre spectral gap). *For every F in the closed domain of $\mathcal{E}^{\mu_{\text{Gin}}}$,*

$$(6) \quad \boxed{\text{Var}_{\mu_{\text{Gin}}}(F) \leq \mathcal{E}^{\mu_{\text{Gin}}}(F, F).}$$

Consequently the canonical closed Ginibre Dirichlet form has spectral-gap rate at least one, in the normalization (5).

Remark 1.3 (Sharp constant). *The proof gives the lower bound one. Smooth linear statistics at very low spatial frequency give the upper bound two from the exact Ginibre structure factor. Thus the present argument locates the optimal infinite-volume rate in $[1, 2]$ but does not identify it.*

2. THE GAUSSIAN $\bar{\partial}$ GAP

Let

$$\gamma_N(\mathrm{d}Z) = \pi^{-N} e^{-|Z|^2} \mathrm{d}A^{\otimes N}(Z)$$

be standard complex Gaussian measure, and let $\mathcal{B}_N \subset L^2(\gamma_N)$ be the Bargmann–Fock subspace of square-integrable entire holomorphic functions. Denote its orthogonal projection by $P_{\mathcal{B}}$.

Lemma 2.1 (Gaussian $\bar{\partial}$ gap). *For every u in the form domain of $\bar{\partial}$,*

$$(7) \quad \|u - P_{\mathcal{B}}u\|_{L^2(\gamma_N)}^2 \leq \sum_{i=1}^N \|\partial_{\bar{z}_i} u\|_{L^2(\gamma_N)}^2.$$

Proof. Expand u in the tensor-product complex Hermite basis. The operator $\bar{\partial}^* \bar{\partial}$ has eigenvalues $|\mathbf{q}|$ on the basis vector of antiholomorphic degree $\mathbf{q}$. Its zero eigenspace is precisely $\mathcal{B}_N$, and every nonzero eigenvalue is at least one. The spectral theorem gives (7). $\square$

Let

$$\Delta_N(Z) = \prod_{1 \leq i < j \leq N} (z_i - z_j), \quad D_N = \deg \Delta_N = \frac{N(N-1)}{2}.$$

We shall use two elementary facts:

- (i) the orthogonal projection $P_{\mathcal{B}}$ commutes with coordinate permutations and common rotations;
- (ii) every alternating holomorphic polynomial is divisible by Δ_N , and hence has degree at least D_N .

3. PROOF OF THE FINITE THEOREM

Normalize so that

$$(8) \quad \mathrm{d}\mu_N = \frac{|\Delta_N|^2}{\|\Delta_N\|_{L^2(\gamma_N)}^2} \mathrm{d}\gamma_N.$$

For a real symmetric f set

$$(9) \quad \psi = \Delta_N f.$$

Then ψ is alternating. Moreover,

$$(10) \quad \int f \mathrm{d}\mu_N = \frac{\langle \psi, \Delta_N \rangle_{L^2(\gamma_N)}}{\|\Delta_N\|_{L^2(\gamma_N)}^2},$$

and, since Δ_N is holomorphic,

$$(11) \quad \sum_{i=1}^N \|\partial_{\bar{z}_i} \psi\|_{L^2(\gamma_N)}^2 = \|\Delta_N\|_{L^2(\gamma_N)}^2 \int \sum_{i=1}^N |\partial_{\bar{z}_i} f|^2 \mathrm{d}\mu_N.$$

For real f ,

$$(12) \quad \sum_i |\nabla_i f|^2 = 4 \sum_i |\partial_{\bar{z}_i} f|^2.$$

It remains to prove the following estimate.

Proposition 3.1 (Vandermonde reality estimate). *Suppose f is real and symmetric and has μ_N -mean zero. If $\psi = \Delta_N f$, then*

$$(13) \quad \|\psi\|_{L^2(\gamma_N)}^2 \leq 2 \sum_{i=1}^N \|\partial_{\bar{z}_i} \psi\|_{L^2(\gamma_N)}^2.$$

Proof. Let $R_\theta f(Z) = f(e^{i\theta} Z)$ and decompose f into common-phase modes,

$$(14) \quad f = \sum_{m \in \mathbb{Z}} f_m, \quad R_\theta f_m = e^{im\theta} f_m.$$

Because f is real,

$$(15) \quad f_{-m} = \overline{f_m}.$$

Set $\psi_m = \Delta_N f_m$. Then ψ_m is alternating and has common-rotation character $D_N + m$. The modes are orthogonal in $L^2(\gamma_N)$, and their $\bar{\partial}$ energies are also orthogonal.

We first treat $m < 0$. If $P_{\mathcal{B}}\psi_m$ were nonzero, it would be an alternating holomorphic function with common-rotation character $D_N + m < D_N$. Such a function is a homogeneous alternating polynomial of degree below D_N , which is impossible. Hence

$$P_{\mathcal{B}}\psi_m = 0 \quad (m < 0).$$

For $m = 0$, the Bargmann projection is an alternating homogeneous polynomial of degree D_N , hence a multiple of Δ_N . The mean-zero condition (10) forces that multiple to vanish. Consequently, by Theorem 2.1,

$$(16) \quad \|\psi_m\|_2^2 \leq \|\bar{\partial}\psi_m\|_2^2 \quad (m \leq 0).$$

For each $m \geq 1$, reality gives

$$(17) \quad \|\psi_m\|_2 = \|\psi_{-m}\|_2.$$

Therefore

$$\begin{aligned} \|\psi_m\|_2^2 + \|\psi_{-m}\|_2^2 &= 2\|\psi_{-m}\|_2^2 \\ &\leq 2\|\bar{\partial}\psi_{-m}\|_2^2 \\ &\leq 2\left(\|\bar{\partial}\psi_m\|_2^2 + \|\bar{\partial}\psi_{-m}\|_2^2\right). \end{aligned}$$

Summing over m , using phase orthogonality, proves (13). $\square$

Proof of Theorem 1.1. Subtract the mean of f . By (8), Theorem 3.1, (11), and (12),

$$\begin{aligned} \text{Var}_{\mu_N}(f) &= \frac{\|\Delta_N f\|_2^2}{\|\Delta_N\|_2^2} \\ &\leq 2 \int \sum_i |\partial_{\bar{z}_i} f|^2 d\mu_N \\ &= \frac{1}{2} \int \sum_i |\nabla_i f|^2 d\mu_N = \mathcal{E}_N(f, f). \end{aligned}$$

The proof is first for smooth functions and extends to the closed form domain by approximation.

For sharpness, introduce the normalized center coordinate

$$S_N = N^{-1/2} \sum_{i=1}^N z_i.$$

An orthogonal linear change of coordinates separates S_N from the relative coordinates. Since Δ_N depends only on differences, S_N is an independent standard complex Gaussian under μ_N . Thus $f = \text{Re } S_N$ satisfies

$$\text{Var}_{\mu_N}(f) = \frac{1}{2}, \quad \mathcal{E}_N(f, f) = \frac{1}{2}.$$

Hence the rate one is optimal. $\square$

4. PASSAGE TO THE INFINITE GINIBRE FIELD

The finite process associated with (1) is determinantal with kernel

$$(18) \quad K_N(z, w) = \frac{1}{\pi} e^{-(|z|^2 + |w|^2)/2} \sum_{k=0}^{N-1} \frac{(z\bar{w})^k}{k!}.$$

The kernels converge, with all derivatives uniformly on compact subsets, to (4). Equivalently, the compressed operators converge locally in trace norm, and the finite point processes converge locally in law to μ_{Gin} .

Let the local cylinder class consist of functions

$$(19) \quad F(\gamma) = \Phi(\langle \varphi_1, \gamma \rangle, \dots, \langle \varphi_M, \gamma \rangle),$$

where $\Phi \in C_b^\infty(\mathbb{R}^M)$ and $\varphi_a \in C_c^\infty(\mathbb{C}; \mathbb{R})$. Its carré du champ is

$$(20) \quad \Gamma(F)(\gamma) = \sum_{x \in \gamma} \left| \sum_{a=1}^M \partial_a \Phi \nabla \varphi_a(x) \right|^2.$$

If K contains all supports involved, then

$$(21) \quad 0 \leq \Gamma(F)(\gamma) \leq C_F \gamma(K).$$

For the finite and infinite Ginibre processes, compact-set counts have uniform exponential moments. Indeed, a determinantal count is a sum of independent Bernoulli variables, and

$$(22) \quad \sup_N \mathbb{E} e^{t\Xi_N(K)} \leq \exp \left((e^t - 1) \frac{|K|}{\pi} \right), \quad t \geq 0,$$

using $K_N(z, z) \leq \pi^{-1}$; see [1]. Hence (21) is uniformly integrable.

Let

$$F_N(z_1, \dots, z_N) = F \left(\sum_{i=1}^N \delta_{z_i} \right).$$

This is a smooth symmetric function. Applying Theorem 1.1,

$$(23) \quad \text{Var}_{\mu_N}(F_N) \leq \frac{1}{2} \mathbb{E}_{\mu_N} \Gamma(F)(\Xi_N).$$

Local determinantal convergence gives convergence of the variances, while local convergence plus (22) gives convergence of the carré-du-champ expectations. Sending $N \rightarrow \infty$ in (23) yields

$$(24) \quad \text{Var}_{\mu_{\text{Gin}}}(F) \leq \frac{1}{2} \int \Gamma(F) d\mu_{\text{Gin}} = \mathcal{E}^{\mu_{\text{Gin}}}(F, F)$$

for every smooth bounded local cylinder F .

Proof of Theorem 1.2. The local cylinder pre-form is closable for μ_{Gin} , and the unlabeled Ginibre diffusion is associated with its canonical closure [2, 3]. Smooth bounded local cylinders are form-dense by truncation and local approximation. Inequality (24) is stable under convergence in the form norm. It therefore extends from the defining core to the full closed domain. $\square$

5. CONSEQUENCES AND COMPARISON

The theorem gives exponential L^2 relaxation:

$$(25) \quad \|P_t F - \mu_{\text{Gin}}(F)\|_{L^2(\mu_{\text{Gin}})} \leq e^{-t} \|F - \mu_{\text{Gin}}(F)\|_{L^2(\mu_{\text{Gin}})}$$

for the semigroup corresponding to the convention (5).

This differs sharply from the one-dimensional sine₂/infinite Dyson diffusion, for which Suzuki proved absence of a spectral gap by a linear-statistics construction [5]. In Ginibre, the quadratic zero of the structure factor makes that obstruction disappear; the theorem above upgrades this first-chaos compatibility to the full nonlinear configuration-space inequality.

The positive gap is also consistent with Osada's zero tagged self-diffusivity theorem [4]. The tagged coordinate is not an $L^2(\mu_{\text{Gin}})$ local observable on the unlabeled configuration space, and the tagged corrector problem is an affine variational problem under a reduced Palm measure. Thus a positive unlabeled spectral gap does not force a positive tagged self-diffusion coefficient.

Remark 5.1 (Status and provenance). *The finite Vandermonde- $\bar{\partial}$ argument and the thermodynamic-limit passage are self-contained modulo standard facts about determinantal convergence and the established construction/closability of the Ginibre Dirichlet form. A targeted search did not locate this precise symmetric finite- N inequality or its infinite-volume consequence. This is a novelty-search statement, not a claim of exhaustive absence from the literature. The proof should receive an independent audit before publication-level certification.*

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