REMARKS ON THE LOGARITHMIC SOBOLEV INEQUALITY FOR THE KURAMOTO MODEL

In this note, we show that the Kuramoto model,

q:Kuramoto}

$$\mathsf{g}(x) = \cos(x), \ x \in \mathbb{T} \qquad \mathbb{P}_{N,\beta} := \frac{1}{Z_{N,\beta}} e^{\frac{\beta}{2N}\sum_{1\le i,j\le N} \mathsf{g}(x_i - x_j)} d\underline{x}_N, \tag{0.1}$$

satisfies a uniform-in-N LSI for inverse temperature $\beta < 2$ by a close inspection of the method of Bauerschmidt-Bodineau [BB19] for spin systems. Here, $\mathbb{T}$ is the circle, which we identify with $[-\pi, \pi)$ under periodic boundary conditions. Our main result is the following theorem.

hm:Kuramoto

Theorem 0.1. *Let $N \ge 2$, $\beta < 2$ and $\mathbb{P}_{N,\beta}$ be as in* (0.1). *Then there exist a constant $C_{LS} > 0$, which depends on β but is independent of N, such that for any test function φ with $\mathbb{E}_{\mu_{N,\beta}}[\varphi^2] = 1$,*

$$\mathrm{Ent}_{\mu_{N,\beta}}(\varphi^2) \le C_{LSI}\mathbb{E}_{\mu_{N,\beta}}\left[|\nabla\varphi|^2\right]. \tag{0.2}$$

Remark 0.2. The condition $\beta < 2$ is sharp, in the sense that a uniform-in-N LSI cannot hold for $\beta \ge 2$. The way to show this is by considering the contrapositive statement that uniqueness of mean-field equilibria. Indeed, it is known that for $\beta \ge 2$, the uniform distribution is *not* the unique solution of the mean-field equation

eq:KurMFeq}

$$\mu_\beta = \frac{1}{Z_\beta} e^{-\frac{\beta}{2}\mathsf{g}*\mu_\beta}. \tag{0.3}$$

Since any solution of (0.3) is a stationary solution of the . A uniform-in-N LSI implies uniform-in-time propagation of chaos .

The relationship between uniform-in-N LSI, uniqueness of mean-field equilibria, and uniform-in-time propagation of chaos has been at least folklore for some years. Details for a class of models, including Kuramoto, are spelled out in [].

Remark 0.3. Since the LSI is preserved under taking marginals, Theorem 0.1 implies that for any $N \ge 2$ and $1 \le k \le N$, the k-point marginal $\mu_{N;k,\beta}$ satisfies an LSI with the same constant C_{LSI}.

Remark 0.4. Since the uniform measure on $\mathbb{T}$ satisfies an LSI with constant ? and $|\cos| \le 1$, it follows from the Holley-Stroock perturbation lemma that $\mu_{N,\beta}$ satisfies an LSI with constant $C_{LS} \le$. The nontriviality of Theorem 0.1 is going from an $O(1)$ upper bound to an $O(1)$ upper bound.

For each $1 \le i \le N$, set $\sigma_i := (\cos(x_i), \sin(x_i)) \in \mathbb{R}^2$. Fix i, j, and observe that

$$\frac{\beta}{N}\cos(x_i - x_j) = M_{ij}\sigma_i \cdot \sigma_j, \qquad M_{ij} := \frac{\beta}{N} I_{2\times 2}, \tag{0.4}$$

where $\cdot$ denotes the standard inner product on $\mathbb{R}^2$. This follows from the formula $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$. Letting M be the $N \times N$ block matrix where the (i,j) entry is the 2×2 matrix M_{ij}, we see that

$$\mu_{N,\beta} = \frac{1}{Z_{N,\beta}} e^{-\frac{1}{2}\langle M\underline{\sigma}_N, \underline{\sigma}_N\rangle} d\underline{x}_N = \frac{1}{Z_{N,\beta}} e^{-\frac{1}{2}\sum_{i,j=1}^N M_{ij}\sigma_i\cdot\sigma_j} d\underline{x}_N, \tag{0.5}$$

with the notation $\underline{\sigma}_N := (\sigma_1, \ldots, \sigma_N)$.

Let λ_+ and λ_- denote the largest and smallest eigenvalues of M, respectively. We set

$$\langle M\rangle := \lambda_+ - \lambda_-. \tag{0.6}$$

Since $\mathbb{M}$ is symmetric and all the diagonal entries of M equal $\frac{\beta}{N}$, it follows from the Gershgorin circle theorem that any eigenvalue λ of M must satisfy

$$-\beta + \frac{2\beta}{N} = \frac{\beta}{N} - \frac{\beta(N-1)}{N} \le \lambda \le \frac{\beta}{N} + \frac{\beta(N-1)}{N} = \beta. \tag{0.7}$$

Hence, $\lambda_+ \le \beta$ and $\lambda_- \ge -\beta(\frac{2}{N}-1)$. We can actually do better for a lower bound for λ_- since M is positive semi-definite. To see this, first observe that for the i-th standard basis vector $e_i \in \mathbb{R}^{2N}$,

$$Me_i = \frac{\beta}{N}\begin{cases}\vec{a}, & i \text{ odd}\\ \vec{b}, & i \text{ even},\end{cases} \tag{0.8}$$

where $\vec{a} = (1,0,\ldots,1,0)$ and $\vec{b} = (0,1,\ldots,0,1)$. Hence, for any $\underline{z}_N \in (\mathbb{R}^2)^N$, with $z_i = (z_i^1, z_i^2)$, we have

$$M\underline{z}_N = \frac{\beta}{N}\left(\sum_{i=1}^N z_i^1\right)\vec{a} + \frac{\beta}{N}\left(\sum_{i=1}^N z_i^2\right)\vec{b}. \tag{0.9}$$

This implies that

$$\langle M\underline{z}_N, \underline{z}_N\rangle = \sum_{j=1}^N z_j^1 \frac{\beta}{N}\left(\sum_{i=1}^N z_i^1\right) + \sum_{j=1}^N z_j^2 \frac{\beta}{N}\left(\sum_{i=1}^N z_i^2\right) = \frac{\beta}{N}\left(\sum_{i=1}^N z_i^1\right)^2 + \frac{\beta}{N}\left(\sum_{i=1}^N z_i^2\right)^2 \ge 0. \tag{0.10}$$

In fact, one also has that $\lambda_+ = \beta$ (i.e., the Gershgorin bound is sharp), since $\vec{a}, \vec{b}$ are eigenvectors of M associated to the eigenvalue β. So, we have shown that $\langle M\rangle = \beta$.

Before proceeding further, we state two preliminary lemmas that we need below. Lemma 0.5 is the $n=2$ specialization (i.e., replace $\mathbb{R}^2$ with $\mathbb{R}^n$ and S^1 with S^{n-1}) of a result from []. Strictly speaking, [] only considers the $n>2$ case, but as noted by Bauerschmidt-Bodineau, the proof extends to the $n=2$ case and is, in fact, simpler. Lemma 0.6 is a special case of [, Theorem D.2].

lem:LSInuh **Lemma 0.5.** *For any $h \in \mathbb{R}^2$, let ν^h be the probability measure on S^1 which is $\propto e^{h\cdot\sigma}d\sigma$. Then ν^h satisfies an LSI with constant $\frac{2}{\gamma}$, which is independent of h: for any test function φ,*

$$\mathrm{Ent}_{\nu^h}(\varphi^2) \le \frac{2}{\gamma}\mathbb{E}_{\nu^h}\left(|\nabla\varphi|^2\right). \tag{0.11}$$

lem:Var **Lemma 0.6.** *For any $h \in \mathbb{R}^2$ and unit vector $x \in \mathbb{R}^2$, one has*

$$\mathrm{Var}_{\nu^h}(x\cdot\sigma) \le \frac{1}{2}, \tag{0.12}$$

where σ is understood as a random variable distributed according to μ^h.

We can make the qualitative assumption that M is strictly positive definite. Let $\delta > 0$, and set $M^\delta := M + \delta I_{2N\times 2N}$. Then $\mathbb{M}^\delta < cI_{2N\times 2N}$ with $c = \beta + 2\delta$. Since the inverse of a positive definite matrix is again positive definite, there exists a positive definite matrix B such that

{eq:Mdinv}
$$(M^\delta)^{-1} = c^{-1}I_{2N\times 2N} + B^{-1}. \tag{0.13}$$

Now since the convolution of two Gaussian measures with covariances A_1, A_2 is the Gaussian measure with covariance $A_1 + A_2$, it follows from (0.13) that

q:Gausconv}
$$\forall \underline{z}_N \in (\mathbb{R}^2)^N, \qquad e^{-\frac{1}{2}\langle M^\delta \underline{z}_N, \underline{z}_N\rangle} = C\int_{(\mathbb{R}^2)^N} e^{-\frac{c}{2}|\underline{w}_N - \underline{x}_N|^2} e^{-\frac{1}{2}\langle \underline{w}_N, B\underline{w}_N\rangle} d\underline{w}_N. \tag{0.14}$$

Letting $\mu^\delta_{N,\beta}$ be the Gibbs measure with M replaced by $\mathbb{M}^\delta$, by Fubini-Tonelli and the identity (0.14), we see that for any test function $\Phi : \mathbb{T}^N \to \mathbb{R}$,

q:EmuNbeFT}
$$\mathbb{E}_{\mu^\delta_{N,\beta}}[\Phi] = \frac{1}{Z_{N,\beta}} \int_{(\mathbb{R}^2)^N} d\underline{w}_N e^{-\frac{1}{2}\langle \underline{w}_N, B\underline{w}_N\rangle} \int_{(\mathbb{T})^N} \Phi(\underline{x}_N) e^{-\frac{c}{2}|\underline{w}_N - \underline{\sigma}_N|^2} d\underline{x}_N. \tag{0.15}$$

Following the terminology of Bauerschmidt-Bodineau, we introduce the *renormalized single-site potential*

{eq:Vdef}
$$V(w) = -\log\left(\int_{\mathbb{T}} e^{-\frac{c}{2}|w-\sigma|^2} dx\right), \tag{0.16}$$

the single-site and multi-site *fluctuation measures* respectively given by

$$\nu^\delta_w := e^{V(w) - \frac{c}{2}|w-\sigma|^2} dx, \quad \forall w \in \mathbb{R}^2, \qquad \nu^\delta_{\underline{w}_N} := \bigotimes_{i=1}^N \nu^\delta_{w_i}, \quad \forall \underline{w}_N \in (\mathbb{R}^2)^N, \tag{0.17}$$

and the *renormalized measure* on $(\mathbb{R}^2)^N$

$$\mu^\delta_{r,N,\beta} := \frac{1}{Z_{r,N,\beta}} e^{-\frac{1}{2}\langle \underline{w}_N, B\underline{w}_N\rangle - \sum_{i=1}^N V(w_i)} d\underline{w}_N \tag{0.18}$$

With this notation, we quickly see from (0.15) that

eq:Expiter}
$$\mathbb{E}_{\mu^\delta_{N,\beta}}[\Phi] = \mathbb{E}_{\mu^\delta_{r,N,\beta}}\left[\mathbb{E}_{\nu^\delta_{\underline{w}_N}}[\Phi]\right]. \tag{0.19}$$

Since $|\sigma| = 1$, we observe that

{eq:expid}
$$e^{-\frac{c}{2}|w-\sigma|^2} = e^{-\frac{c}{2}|w|^2} e^{cw\cdot\sigma}, \tag{0.20}$$

which implies that $\nu^\delta_w = \nu^{cw}$ (recalling the notation from Lemma 0.5).

Next, we show that the renormalized measure $\mu^\delta_{r,N,\beta}$ satisfies an LSI. To this end, we first claim that the renormalized single-site potential V is uniformly convex. To see this, observe from (0.16) and (0.20) that

$$V(w) = \frac{c}{2}|w|^2 - \log\left(\int_{\mathbb{T}} e^{cw\cdot\sigma} dx\right). \tag{0.21}$$

Computing the first and second derivatives of V,

{eq:Vfd}
$$\partial_a V(w) = cw^a - \frac{\int_{\mathbb{T}} c\sigma^a e^{cw\cdot\sigma} dx}{\int_{\mathbb{T}} e^{cw\cdot\sigma} dx} \tag{0.22}$$

{eq:Vsd}
$$\begin{aligned} \partial_a\partial_b V(w) &= c\delta_{ab} - \frac{\int_{\mathbb{T}} c^2\sigma^a\sigma^b e^{cw\cdot\sigma} dx}{\int_{\mathbb{T}} e^{cw\cdot\sigma} dx} + \frac{\left(\int_{\mathbb{T}} c\sigma^a e^{cw\cdot\sigma} dx\right)\left(\int_{\mathbb{T}} c\sigma^b e^{cw\cdot\sigma} dx\right)}{\left(\int_{\mathbb{T}} e^{cw\cdot\sigma} dx\right)^2} \\ &= c\delta_{ab} - c^2 \int_{\mathbb{T}} \sigma^a\sigma^b d\nu^{cw}(\sigma) + c^2 \left(\int_{\mathbb{T}} \sigma^a d\nu^{cw}(\sigma)\right)\left(\int_{\mathbb{T}} \sigma^b d\nu^{cw}(\sigma)\right). \end{aligned} \tag{0.23}$$

So, for any $z \in \mathbb{R}^2$,

$$\begin{aligned} \mathrm{Hess}(V)(w)z\cdot z &= c\delta_{ab}z^a z^b - c^2 \int_{\mathbb{T}} z^a z^b \sigma^a\sigma^b d\nu^{cw}(\sigma) + c^2\left(\int_{\mathbb{T}} z^a\sigma^a d\nu^{cw}(\sigma)\right)\left(\int_{\mathbb{T}} z^b\sigma^b d\nu^{cw}(\sigma)\right) \\ &= c|z|^2 - c^2\mathbb{E}_{\nu^{cw}}\left[(\sigma\cdot z)^2\right] + c^2\left(\mathbb{E}_{\nu^{cw}}[\sigma\cdot z]\right)^2 \\ &= c|z|^2 - c^2\mathrm{Var}_{\nu^{cw}}[\sigma\cdot z] \\ &\geq \left(c - \frac{c^2}{2}\right)|z|^2, \end{aligned} \tag{0.24}$$

where to obtain the ultimate line, we used Lemma 0.6. Recalling that $c = \beta + 2\delta$, we see that

$$c - \frac{c^2}{2} = \frac{(\beta+2\delta)(2-\beta-2\delta)}{2}. \tag{0.25}$$

So, provided $\beta < 2$, we may choose $\delta > 0$ sufficiently small so that $\beta + 2\delta < 2$. Under such assumptions, the Bakry-Emery criterion [] is satisfied (recall that B is positive definite and its contribution may be discarded), and the probability measure $\mu^{\delta}_{r,N,\beta}$ satisfies an LSI with constant $\frac{4}{2c-c^2}$. We also note that the probability measure $\nu^{\delta}_{\underline{w}_N}$ satisfies an LSI with constant $\frac{2}{\gamma}$. This is a consequence of Lemma 0.5 and the fact that the LSI tensorizes.

Unpacking the definition of the entropy and using the identity (0.19), we write

$$\begin{aligned}
\mathrm{Ent}_{\mu^{\delta}_{N,\beta}}\left[\Phi^2\right] &= \mathbb{E}_{\mu^{\delta}_{N,\beta}}\left[\Phi^2\log\Phi^2\right] - \mathbb{E}_{\mu^{\delta}_{N,\beta}}\left[\Phi^2\right]\log\mathbb{E}_{\mu^{\delta}_{N,\beta}}\left[\Phi^2\right] \\
&= \mathbb{E}_{\mu^{\delta}_{r,N,\beta}}\left[\mathbb{E}_{\nu^{\delta}_{\underline{w}_N}}\left[\Phi^2\log\Phi^2\right]\right] - \mathbb{E}_{\mu^{\delta}_{r,N,\beta}}\left[\mathbb{E}_{\nu^{\delta}_{\underline{w}_N}}\left[\Phi^2\right]\log\mathbb{E}_{\nu^{\delta}_{\underline{w}_N}}\left[\Phi^2\right]\right] \\
&\quad + \mathbb{E}_{\mu^{\delta}_{r,N,\beta}}\left[\mathbb{E}_{\nu^{\delta}_{\underline{w}_N}}\left[\Phi^2\right]\log\mathbb{E}_{\nu^{\delta}_{\underline{w}_N}}\left[\Phi^2\right]\right] - \mathbb{E}_{\mu^{\delta}_{r,N,\beta}}\left[\mathbb{E}_{\nu^{\delta}_{\underline{w}_N}}\left[\Phi^2\right]\right]\log\mathbb{E}_{\mu^{\delta}_{r,N,\beta}}\left[\mathbb{E}_{\nu^{\delta}_{\underline{w}_N}}\left[\Phi^2\right]\right] \\
&= \mathbb{E}_{\mu^{\delta}_{r,N,\beta}}\left[\mathrm{Ent}_{\nu^{\delta}_{\underline{w}_N}}(\Phi^2)\right] + \mathrm{Ent}_{\mu^{\delta}_{r,N,\beta}}\left(\mathbb{E}_{\nu^{\delta}_{\underline{w}_N}}\left[\Phi^2\right]\right). \qquad (0.26)
\end{aligned}$$

To lighten the notation, set $\Psi(\underline{w}_N) := \left(\mathbb{E}_{\nu^{\delta}_{\underline{w}_N}}\left[\Phi^2\right]\right)^{1/2}$. Applying the LSI for $\nu^{\delta}_{\underline{w}_N}$, we obtain

$$\mathbb{E}_{\mu^{\delta}_{r,N,\beta}}\left[\mathrm{Ent}_{\nu^{\delta}_{\underline{w}_N}}(\Phi^2)\right] \le \frac{2}{\gamma}\mathbb{E}_{\mu^{\delta}_{r,N,\beta}}\left[\mathbb{E}_{\nu^{\delta}_{\underline{w}_N}}\left[|\nabla\Phi|^2\right]\right] = \frac{2}{\gamma}\mathbb{E}_{\mu^{\delta}_{N,\beta}}\left[|\nabla\Phi|^2\right], \tag{0.27}$$

where the final equality is by (0.19), and applying the LSI for $\mu^{\delta}_{r,N,\beta}$, we obtain

$$\mathrm{Ent}_{\mu^{\delta}_{r,N,\beta}}\left(\Psi^2\right) \le \frac{4}{2c-c^2}\mathbb{E}_{\mu^{\delta}_{r,N,\beta}}\left[|\nabla\Psi|^2\right]. \tag{0.28}$$

For any $1 \le i \le N$, the chain rule and formula (0.22) yield

q:gradPsi1}

$$\begin{aligned}
\nabla_{w_i}\Psi(\underline{w}_N) &= \frac{\nabla_{w_i}\mathbb{E}_{\nu^{\delta}_{\underline{w}_N}}\left[\Phi^2\right]}{2\Psi(\underline{w}_N)} \\
&= \frac{1}{2\Psi(\underline{w}_N)}\int_{\mathbb{T}^N}\left[\nabla V(w_i) - c(w-\sigma)_i\right]\Phi^2(\underline{x}_N)d\nu^{\delta}_{\underline{w}_N}(\underline{x}_N) \\
&= \frac{1}{2\Psi(\underline{w}_N)}\int_{\mathbb{T}^N}\left[cw_i - c\mathbb{E}_{\nu^{cw_i}}[\sigma^i] - c(w-\sigma_i)\right]\Phi^2(\underline{x}_N)d\nu^{\delta}_{\underline{w}_N}(\underline{x}_N) \\
&= \frac{c}{2\Psi(\underline{w}_N)}\int_{\mathbb{T}^N}\left[\sigma_i - \mathbb{E}_{\nu^{cw_i}}[\sigma_i]\right]\Phi^2(\underline{x}_N)d\nu^{\delta}_{\underline{w}_N}(\underline{x}_N) \\
&= \frac{c}{2\Psi(\underline{w}_N)}\int_{(\mathbb{T}^N)^2}\left[\sigma_i - \sigma_i'\right]\Phi^2(\underline{x}_N)d(\nu^{\delta}_{\underline{w}_N})^{\otimes 2}(\underline{x}_N,\underline{x}_N') \\
&= \frac{c}{4\Psi(\underline{w}_N)}\int_{(\mathbb{T}^N)^2}\left[\sigma_i - \sigma_i'\right]\left[\Phi^2(\underline{x}_N) - \Phi^2(\underline{x}_N')\right]d(\nu^{\delta}_{\underline{w}_N})^{\otimes 2}(\underline{x}_N,\underline{x}_N'), \qquad (0.29)
\end{aligned}$$

where the penultimate line is by Fubini-Tonelli with $\bigotimes_{i=1}^N \nu^{cw_i} = \nu^{\delta}_{\underline{w}_N}$ and the ultimate line is by symmetrization. Above, we have introduced the notation $\sigma_i' := (\cos(x_i'), \sin(x_i'))$. Factoring

$\Phi^2(\underline{x}_N) - \Phi^2(\underline{x}'_N)$ and then applying Cauchy-Schwarz,

$$\begin{aligned}
&\left| \int_{\mathbb{T}^N} \left[\sigma_i - \sigma_i'\right] \left[\Phi^2(\underline{x}_N) - \Phi^2(\underline{x}'_N)\right] d(\nu^{\delta}_{\underline{w}_N})^{\otimes 2}(\underline{x}_N, \underline{x}'_N) \right| \\
&\leq \left(\int_{(\mathbb{T}^N)^2} \left|\Phi(\underline{x}_N) - \Phi(\underline{x}'_N)\right|^2 d(\nu^{\delta}_{\underline{w}_N})^{\otimes 2}(\underline{x}_N, \underline{x}'_N) \right)^{\frac{1}{2}} \\
&\quad \times \left(\int_{(\mathbb{T}^N)^2} |\sigma_i - \sigma_i'|^2 |\Phi(\underline{x}_N) + \Phi(\underline{x}'_N)|^2 d(\nu^{\delta}_{\underline{w}_N})^{\otimes 2}(\underline{x}_N, \underline{x}'_N) \right)^{\frac{1}{2}}
\end{aligned}$$

q:gradPsi2}

$$\leq 4 \left(\mathrm{Var}_{\nu^{\delta}_{\underline{w}_N}} \left[\Phi^2\right] \right)^{1/2} \left(\mathbb{E}_{\nu^{\delta}_{\underline{w}_N}} \left[\Phi^2\right] \right)^{1/2}, \tag{0.30}$$

where we have implicitly used that $|\sigma_i - \sigma_i'| \leq 2$ and another application of Fubini-Tonelli in obtaining the final line. Recalling the definition of Ψ and combining (0.29) and (0.30), we arrive at

$$\mathbb{E}_{\mu^{\delta}_{r,N,\beta}} \left[|\nabla \Psi|^2\right] \leq \tag{0.31}$$

Since $\nu_{\underline{w}_N}$ satisfies an LSI with constant $\frac{2}{\gamma}$, it also satisfies a Poincaré inequality with constant $\frac{1}{\gamma}$. Hence,

$$\mathrm{Var}_{\nu_{\underline{w}_N}} \left[\Phi^2\right] \leq \frac{1}{\gamma} \mathbb{E}_{\nu_{\underline{w}_N}} \left[|\nabla \Phi|^2\right]. \tag{0.32}$$

Letting $\delta \to 0$ and appealing to dominated convergence, we arrive at the conclusion of Theorem 0.1.

References

[BB19] Roland Bauerschmidt and Thierry Bodineau. A very simple proof of the LSI for high temperature spin systems. *J. Funct. Anal.*, 276(8):2582–2588, 2019.