

PERMUTATION-INVARIANT FUNCTIONAL INEQUALITIES FOR FINITE GINIBRE I: THE EXACT POINCARÉ RATE AND THE CENTERED LSI FRONTIER

TECHNICAL RESEARCH MEMORANDUM

ABSTRACT. For the full labeled complex Ginibre eigenvalue law, the raw Poincaré and logarithmic Sobolev rates degenerate because of a tagged-particle corrector. We begin the permutation-invariant problem independently and prove that this mechanism disappears completely at the linearized level. With the raw Euclidean Dirichlet form, the Poincaré rate restricted to permutation-invariant observables is exactly $2N$ for every N . The proof uses the Vandermonde isometry into antisymmetric Gaussian space, the spectral gap above the Bargmann–Fock projection, and a global-rotation decomposition which forces at least half of the L^2 mass outside the holomorphic subspace. We also classify the extremizers.

We then factor out the Gaussian center of mass exactly. For the logarithmic Sobolev problem, the full symmetric rate equals the centered relative symmetric rate. Moreover, the latter is at most $2N$, and the restriction to centered radial functions has sharp rate exactly $2N$. Using Kostlan’s radial factorization [2], we further prove that the full permutation-invariant sector of observables depending only on the unordered individual radii $(|z_1|, \dots, |z_N|)$ also has sharp logarithmic Sobolev rate $2N$. Thus any logarithmic Sobolev degeneration must be nonlinear and must use angular or genuinely shape-dependent information; neither a perturbative low mode nor any origin-radial statistic can produce it.

1. SETUP AND VERDICT

Let

$$(1) \quad \mu_N(dZ) = \frac{1}{Z_N} e^{-N \sum_{j=1}^N |z_j|^2} \prod_{1 \leq i < j \leq N} |z_i - z_j|^2 dZ, \quad Z = (z_1, \dots, z_N) \in \mathbb{C}^N.$$

For real-valued F in the closed Sobolev form domain, set

$$(2) \quad \mathcal{E}_N(F) := \int_{\mathbb{C}^N} \sum_{j=1}^N |\nabla_{z_j} F|^2 d\mu_N.$$

The raw Poincaré and logarithmic Sobolev rates are normalized by

$$(3) \quad \text{Var}_{\mu_N}(F) \leq \frac{1}{\lambda_N^{\text{PI}}} \mathcal{E}_N(F),$$

$$(4) \quad \text{Ent}_{\mu_N}(F^2) \leq \frac{2}{\lambda_N^{\text{LSI}}} \mathcal{E}_N(F).$$

Let $H_{\text{sym}}^1(\mu_N)$ be the closed subspace of permutation-invariant functions, and define $\lambda_{N,\text{sym}}^{\text{PI}}$ and $\lambda_{N,\text{sym}}^{\text{LSI}}$ by restricting the corresponding variational problems to this subspace.

The first audit target has the following exact resolution.

Theorem 1.1 (Exact symmetric Poincaré rate). *For every $N \geq 1$,*

$$(5) \quad \boxed{\lambda_{N,\text{sym}}^{\text{PI}} = 2N.}$$

The nonconstant extremizers are precisely

$$(6) \quad F(Z) = \operatorname{Re} \left(c \sum_{j=1}^N z_j \right) + a, \quad c \in \mathbb{C} \setminus \{0\}, \quad a \in \mathbb{R}.$$

The proof below is independent of the tagged-corrector argument. In particular, no finite- or infinite-volume Palm input is used.

2. THE VANDERMONDE GROUND-STATE TRANSFORM

Write

$$(7) \quad \Delta_N(Z) := \prod_{1 \leq i < j \leq N} (z_i - z_j), \quad D_N := \frac{N(N-1)}{2},$$

and let

$$(8) \quad \gamma_N(dZ) := \left(\frac{N}{\pi} \right)^N e^{-N \sum_{j=1}^N |z_j|^2} dZ.$$

Choose $a_N > 0$ so that

$$(9) \quad d\mu_N = a_N^2 |\Delta_N|^2 d\gamma_N.$$

Multiplication by the Vandermonde defines an isometry

$$(10) \quad U_N : L_{\text{sym}}^2(\mu_N) \longrightarrow L_{\text{asym}}^2(\gamma_N), \quad U_N F = a_N \Delta_N F.$$

Here “asym” means antisymmetric under coordinate permutations. Surjectivity holds at the L^2 level because division by Δ_N is defined away from the collision set, which has Lebesgue and Gaussian measure zero.

Use the convention

$$(11) \quad \bar{\partial}_j = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j}), \quad z_j = x_j + iy_j.$$

Since Δ_N is holomorphic, for every real-valued smooth symmetric F ,

$$(12) \quad \bar{\partial}_j(U_N F) = a_N \Delta_N \bar{\partial}_j F.$$

Since $4|\bar{\partial}_j F|^2 = |\nabla_{z_j} F|^2$ for real F , this gives the exact form identity

$$(13) \quad \boxed{\mathcal{E}_N(F) = 4 \sum_{j=1}^N \|\bar{\partial}_j(U_N F)\|_{L^2(\gamma_N)}^2.}$$

It first holds on the smooth symmetric core and then extends to the closed form domain by closability.

3. THE FOCK GAP ABOVE THE HOLOMORPHIC PROJECTION

Let $\mathcal{B}_N$ denote the orthogonal projection from $L^2(\gamma_N)$ onto the Bargmann–Fock space of square-integrable entire functions.

Lemma 3.1 (Fock gap). *For every G in the closed $\bar{\partial}$ form domain,*

$$(14) \quad \sum_{j=1}^N \|\bar{\partial}_j G\|_2^2 \geq N \|(I - \mathcal{B}_N)G\|_2^2.$$

Equality holds precisely when $(I - \mathcal{B}_N)G$ belongs to the first anti-holomorphic Gaussian chaos.

Proof. In one complex coordinate with Gaussian weight $(N/\pi)e^{-N|z|^2}$, the complex Hermite decomposition diagonalizes $\bar{\partial}^*\bar{\partial}$ with eigenvalues Nq , $q \in \{0, 1, 2, \dots\}$, where the $q = 0$ eigenspace is the holomorphic Fock space. Tensorizing over the N coordinates yields

$$(15) \quad L^2(\gamma_N) = \bigoplus_{q \geq 0} \mathcal{H}_q, \quad \sum_j \|\bar{\partial}_j G\|_2^2 = N \sum_{q \geq 1} q \|G_q\|_2^2.$$

Since $\mathcal{B}_N G = G_0$, the claim follows. $\square$

The Fock gap alone is useless on the full antisymmetric Gaussian space, because that space contains infinitely many holomorphic functions. The real-valued quotient constraint is what removes enough of the holomorphic mass.

4. THE ROTATION-PROJECTION ESTIMATE

Let $R_\theta Z = e^{i\theta} Z$. The Ginibre law is invariant under R_θ . Every $F \in L^2(\mu_N)$ has the orthogonal Fourier decomposition

$$(16) \quad F = \sum_{m \in \mathbb{Z}} F_m, \quad F_m(R_\theta Z) = e^{im\theta} F_m(Z).$$

If F is real-valued, then

$$(17) \quad F_{-m} = \overline{F_m}, \quad \|F_{-m}\|_2 = \|F_m\|_2.$$

Set $G_m = U_N F_m$. Since $\Delta_N(R_\theta Z) = e^{iD_N\theta} \Delta_N(Z)$, the Gaussian function G_m has rotation charge $D_N + m$.

Lemma 4.1 (At most half of the mass is holomorphic). *Let $F \in L^2_{\text{sym}}(\mu_N)$ be real-valued with $\int F d\mu_N = 0$, and let $G = U_N F$. Then*

$$(18) \quad \|\mathcal{B}_N G\|_2^2 \leq \frac{1}{2} \|G\|_2^2.$$

Proof. The Bargmann projection commutes with rotations and permutations. Thus $\mathcal{B}_N G_m$ is holomorphic, antisymmetric, and has charge $D_N + m$.

Every holomorphic antisymmetric entire function is divisible by Δ_N : antisymmetry forces a zero on each hyperplane $z_i = z_j$, and holomorphic division by the corresponding linear factors may be iterated. Therefore a nonzero holomorphic antisymmetric function has total degree at least D_N . It follows that

$$(19) \quad \mathcal{B}_N G_m = 0 \quad \text{for } m < 0.$$

For $m = 0$, the charge- D_N holomorphic antisymmetric space is $\text{span}\{\Delta_N\}$. Moreover,

$$(20) \quad \langle G_0, a_N \Delta_N \rangle_{L^2(\gamma_N)} = \int F_0 d\mu_N = \int F d\mu_N = 0,$$

so $\mathcal{B}_N G_0 = 0$ as well.

Consequently,

$$(21) \quad \|\mathcal{B}_N G\|_2^2 = \sum_{m > 0} \|\mathcal{B}_N G_m\|_2^2 \leq \sum_{m > 0} \|G_m\|_2^2$$

$$(22) \quad = \sum_{m > 0} \|F_m\|_{L^2(\mu_N)}^2 = \frac{1}{2} \sum_{m \neq 0} \|F_m\|_2^2 \leq \frac{1}{2} \|F\|_2^2 = \frac{1}{2} \|G\|_2^2.$$

$\square$

5. PROOF OF THE EXACT SYMMETRIC POINCARÉ THEOREM

Proof of Theorem 1.1. Let F be real, symmetric, and mean zero, and set $G = U_N F$. Combining (13), Lemma 3.1, and Lemma 4.1,

$$\begin{aligned}
 (23) \quad \mathcal{E}_N(F) &= 4 \sum_{j=1}^N \|\bar{\partial}_j G\|_2^2 \\
 (24) \quad &\geq 4N \|(I - \mathcal{B}_N)G\|_2^2 \\
 (25) \quad &= 4N (\|G\|_2^2 - \|\mathcal{B}_N G\|_2^2) \\
 (26) \quad &\geq 2N \|G\|_2^2 = 2N \|F\|_{L^2(\mu_N)}^2.
 \end{aligned}$$

This proves $\lambda_{N,\text{sym}}^{\text{PI}} \geq 2N$ on the smooth symmetric core and hence on its closure.

For sharpness, take

$$(27) \quad F_c(Z) = \text{Re} \left(c \sum_{j=1}^N z_j \right).$$

The center coordinate $N^{-1/2} \sum_j z_j$ is a centered complex Gaussian with density $(N/\pi) e^{-N|u|^2}$. Hence

$$(28) \quad \text{Var}_{\mu_N}(F_c) = \frac{|c|^2}{2}, \quad \mathcal{E}_N(F_c) = N|c|^2,$$

so the quotient is $2N$.

We finally classify equality. Equality in Lemma 4.1 forces the zero rotation mode to vanish and each positive-charge component G_m to be holomorphic. Hence, for $m > 0$,

$$(29) \quad F_m = S_m(z_1, \dots, z_N),$$

where S_m is a symmetric holomorphic homogeneous polynomial of degree m . Reality gives $F_{-m} = \overline{S_m}$. Equality in the Fock gap forces $G - \mathcal{B}_N G$ to have anti-holomorphic chaos degree one. But

$$(30) \quad G_{-m} = a_N \Delta_N \overline{S_m}$$

can have anti-holomorphic degree at most one only when $m = 1$. The symmetric holomorphic degree-one space is $\text{span}\{\sum_j z_j\}$. Conversely, translation invariance of the Vandermonde gives

$$(31) \quad \mathcal{B}_N \left(\Delta_N \overline{\sum_j z_j} \right) = \frac{1}{N} \sum_{j=1}^N \partial_{z_j} \Delta_N = 0,$$

so these functions do attain equality. Adding constants completes the classification. $\square$

Remark 5.1 (Closed-domain point). The lower bound is first proved for real-valued smooth symmetric functions. The raw Dirichlet form is, by definition, the closure of the smooth Euclidean gradient form in $L^2(\mu_N)$. Averaging a smooth core sequence over the finite permutation group preserves convergence and produces a symmetric core sequence. Therefore no unproved maximal-versus-minimal Sobolev-domain identification is used.

6. EXACT CENTER-OF-MASS REDUCTION

Define

$$(32) \quad u := \frac{1}{\sqrt{N}} \sum_{j=1}^N z_j, \quad y_j := z_j - \frac{u}{\sqrt{N}}, \quad \sum_{j=1}^N y_j = 0.$$

Let

$$(33) \quad \mathcal{H}_N^{\text{rel}} := \left\{ Y \in \mathbb{C}^N : \sum_{j=1}^N y_j = 0 \right\}.$$

The transformation $Z \mapsto (u, Y)$ is unitary, and

$$(34) \quad \sum_j |z_j|^2 = |u|^2 + \sum_j |y_j|^2, \quad \Delta_N(Z) = \Delta_N(Y).$$

Thus

$$(35) \quad \mu_N = \gamma_N^{\text{cm}} \otimes \nu_N^{\text{rel}}, \quad \gamma_N^{\text{cm}}(du) = \frac{N}{\pi} e^{-N|u|^2} du,$$

where ν_N^{rel} is the centered Ginibre law on $\mathcal{H}_N^{\text{rel}}$. Permutations act trivially on u and naturally on Y .

Let $\lambda_{N,\text{rel,sym}}^{\text{PI}}$ and $\lambda_{N,\text{rel,sym}}^{\text{LSI}}$ denote the restricted rates for the relative law.

Proposition 6.1 (Restricted tensorization). *For every N ,*

$$(36) \quad \lambda_{N,\text{sym}}^{\text{PI}} = \min\{2N, \lambda_{N,\text{rel,sym}}^{\text{PI}}\},$$

$$(37) \quad \lambda_{N,\text{sym}}^{\text{LSI}} = \min\{2N, \lambda_{N,\text{rel,sym}}^{\text{LSI}}\}.$$

Proof. The Poincaré identity follows from the conditional variance decomposition, applying the Gaussian Poincaré inequality in u and the restricted relative inequality in Y . The conditional average over u remains permutation invariant in Y , and Jensen controls its relative gradient. Testing functions of one factor alone gives the reverse inequality.

For entropy, write

$$(38) \quad \text{Ent}_{\gamma_N^{\text{cm}} \otimes \nu_N^{\text{rel}}}(F^2) = \int \text{Ent}_{\nu_N^{\text{rel}}}(F(u, \cdot)^2) d\gamma_N^{\text{cm}}(u)$$

$$(39) \quad + \text{Ent}_{\gamma_N^{\text{cm}}} \left(\int F(u, Y)^2 d\nu_N^{\text{rel}}(Y) \right).$$

Set

$$(40) \quad g(u) = \left(\int F(u, Y)^2 d\nu_N^{\text{rel}}(Y) \right)^{1/2}.$$

Then

$$(41) \quad |\nabla_u g(u)|^2 \leq \int |\nabla_u F(u, Y)|^2 d\nu_N^{\text{rel}}(Y)$$

by Cauchy–Schwarz. Apply the two factor inequalities and then test one-factor functions. $\square$

Theorem 1.1 gives

$$(42) \quad \lambda_{N,\text{rel,sym}}^{\text{PI}} \geq 2N.$$

No asymptotic information on the relative LSI follows from this linearized bound.

7. THE CENTERED RADIAL LSI HAS SHARP RATE $2N$

The relative LSI admits a useful exact one-dimensional subproblem. Let

$$(43) \quad S(Y) := \sum_{j=1}^N |y_j|^2.$$

Write

$$(44) \quad \kappa_N := D_N + N - 1 = \frac{(N-1)(N+2)}{2}.$$

Since the relative space has real dimension $2(N-1)$ and $|\Delta_N|^2$ is homogeneous of degree $2D_N$, polar coordinates show that under ν_N^{rel} ,

$$(45) \quad S \sim \text{Gamma}(\kappa_N, \text{rate } N).$$

Equivalently, the radial variable $r = \sqrt{S}$ has density proportional to

$$(46) \quad r^{2\kappa_N-1} e^{-Nr^2} \mathbf{1}_{(0,\infty)}(r) \, dr.$$

Theorem 7.1 (Sharp radial LSI). *For $N \geq 2$, the logarithmic Sobolev rate of ν_N^{rel} restricted to functions of $S(Y)$ alone is exactly $2N$.*

Proof. For a radial function $F(Y) = f(r)$, the relative Dirichlet form is

$$(47) \quad \int_0^\infty |f'(r)|^2 \rho_N(r) \, dr, \quad \rho_N(r) \propto e^{-V_N(r)},$$

where

$$(48) \quad V_N(r) = Nr^2 - (2\kappa_N - 1) \log r + \text{const}.$$

Since

$$(49) \quad V_N''(r) = 2N + \frac{2\kappa_N - 1}{r^2} \geq 2N,$$

the one-dimensional Bakry–Émery criterion gives

$$(50) \quad \text{Ent}_{\rho_N}(f^2) \leq \frac{1}{N} \int_0^\infty |f'|^2 \rho_N \, dr,$$

which is rate $2N$ in the normalization (4).

To see sharpness, for $0 < t < N$ consider the probability density with respect to ν_N^{rel}

$$(51) \quad q_t(Y) = \left(1 - \frac{t}{N}\right)^{\kappa_N} e^{tS(Y)}.$$

Set $x = t/N \in (0, 1)$. Direct Gamma calculations give

$$(52) \quad \text{Ent}_{\nu_N^{\text{rel}}}(q_t) = \kappa_N \left[\frac{x}{1-x} + \log(1-x) \right],$$

$$(53) \quad \mathcal{I}_{\nu_N^{\text{rel}}}(q_t) := \int \frac{|\nabla q_t|^2}{q_t} \, d\nu_N^{\text{rel}} = \frac{4N\kappa_N x^2}{1-x}.$$

Therefore the LSI Rayleigh quotient is

$$(54) \quad \frac{\mathcal{I}(q_t)}{2 \text{Ent}(q_t)} = 2N \frac{x^2}{x + (1-x) \log(1-x)}.$$

It converges to $2N$ as $x \uparrow 1$. Thus the rate cannot exceed $2N$. □

Corollary 7.2 (Exact removal of the center for LSI). *For every $N \geq 2$,*

$$(55) \quad \boxed{\lambda_{N,\text{sym}}^{\text{LSI}} = \lambda_{N,\text{rel},\text{sym}}^{\text{LSI}} \leq 2N.}$$

Proof. The radial test family gives $\lambda_{N,\text{rel},\text{sym}}^{\text{LSI}} \leq 2N$. Insert this into (37). □

8. THE ENTIRE UNORDERED RADIAL SECTOR HAS SHARP LSI RATE $2N$

The preceding theorem treats the total centered radius. Kostlan's radial factorization [2] gives a second exact reduction which rules out every symmetric observable depending only on the individual distances from the origin.

Let

$$(56) \quad \Gamma_{k,N}(\mathrm{d}s) := \frac{N^k}{\Gamma(k)} s^{k-1} e^{-Ns} \mathbf{1}_{(0,\infty)}(s) \mathrm{d}s, \quad k = 1, \dots, N.$$

Lemma 8.1 (Kostlan form identity). *Let $f : (0, \infty)^N \rightarrow \mathbb{R}$ be smooth and symmetric, and set*

$$(57) \quad F(Z) = f(|z_1|^2, \dots, |z_N|^2).$$

If $S_1, \dots, S_N$ are independent with $S_k \sim \Gamma_{k,N}$, then

$$(58) \quad \mathrm{Ent}_{\mu_N}(F^2) = \mathrm{Ent}_{\otimes_{k=1}^N \Gamma_{k,N}}(f^2),$$

$$(59) \quad \mathcal{E}_N(F) = 4 \sum_{k=1}^N \int s_k |\partial_k f(s)|^2 \prod_{j=1}^N \Gamma_{j,N}(\mathrm{d}s_j).$$

Proof. Write $z_i = \sqrt{s_i} e^{i\theta_i}$. Expanding the two determinants in $|\Delta_N|^2$ and integrating the angles gives

$$(60) \quad \int_{[0,2\pi]^N} |\Delta_N(\sqrt{s_1} e^{i\theta_1}, \dots, \sqrt{s_N} e^{i\theta_N})|^2 \mathrm{d}\theta = (2\pi)^N \sum_{\sigma \in S_N} \prod_{i=1}^N s_i^{\sigma(i)-1}.$$

Thus the law of the unordered vector $(|z_1|^2, \dots, |z_N|^2)$ is the symmetrization of the product law $\otimes_{k=1}^N \Gamma_{k,N}$. Every symmetric integrand therefore has the same expectation under these two laws. This gives (58). Moreover,

$$(61) \quad \sum_{i=1}^N |\nabla_{z_i} F|^2 = 4 \sum_{i=1}^N s_i |\partial_i f(s)|^2,$$

whose right-hand side is symmetric because f is symmetric. Applying the same expectation identity proves (59). $\square$

Lemma 8.2 (Sharp weighted Gamma LSI). *For every $k \geq 1$, the measure $\Gamma_{k,N}$ equipped with the Dirichlet form*

$$(62) \quad \mathcal{D}_{k,N}(g) = 4 \int_0^\infty s |g'(s)|^2 \Gamma_{k,N}(\mathrm{d}s)$$

has logarithmic Sobolev rate exactly $2N$.

Proof. Under $s = r^2$, the measure becomes proportional to

$$(63) \quad r^{2k-1} e^{-Nr^2} \mathbf{1}_{(0,\infty)}(r) \mathrm{d}r,$$

and (62) becomes the ordinary one-dimensional gradient form. Its potential has second derivative

$$(64) \quad 2N + \frac{2k-1}{r^2} \geq 2N.$$

Hence the Bakry–Émery criterion gives rate at least $2N$. The normalized exponential tilts

$$(65) \quad q_t(s) = \left(1 - \frac{t}{N}\right)^k e^{ts}, \quad 0 < t < N,$$

have Fisher/entropy quotient tending to $2N$ as $t \uparrow N$, exactly as in (52)–(54). The rate is therefore sharp. $\square$

Theorem 8.3 (Sharp LSI on the unordered radial sector). *For every $N \geq 1$, the logarithmic Sobolev rate of μ_N restricted to real permutation-invariant functions of $(|z_1|, \dots, |z_N|)$ is exactly $2N$.*

Proof. Lemma 8.2 and tensorization give rate $2N$ for the product form on the right side of (59); Lemma 8.1 transfers the inequality to the unordered radial sector. Sharpness follows by taking a function of $\sum_i |z_i|^2$ alone and using the exponential tilts from Theorem 7.1 with the full radial shape parameter $D_N + N = N(N+1)/2$. $\square$

9. WHAT IS NOW RULED OUT, AND WHAT REMAINS

The exact Poincaré theorem and the two sharp radial theorems have five immediate consequences.

- (i) There is no sequence of symmetric observables F_N with order-one variance and $\mathcal{E}_N(F_N) \rightarrow 0$.
- (ii) Linearizing the symmetric LSI around the constant density only sees the scale $2N$. Thus any degeneration of $\lambda_{N,\text{sym}}^{\text{LSI}}$ must be genuinely nonlinear.
- (iii) By Corollary 7.2, the LSI question is entirely a problem on the centered unlabeled relative configuration.
- (iv) Theorem 7.1 rules out the total centered quadratic radius as the nonlinear obstruction.
- (v) Theorem 8.3 rules out every symmetric statistic of the unordered radii about the origin, including origin-centered hole, overcrowding, and extreme-radius profiles. Any counterexample must exploit angular information, angular-radial coupling, relative-to-center geometry, collisions, or another genuinely nonradial shape variable.

The outcome-neutral frontier is therefore

$$(66) \quad \boxed{\text{determine the asymptotics of } \lambda_{N,\text{rel},\text{sym}}^{\text{LSI}}.}$$

There are two competing certificates.

9.1. Positive/nondegeneration certificate. The sharp candidate is

$$(67) \quad \text{Ent}_{\nu_N^{\text{rel}}}(F^2) \leq \frac{1}{N} \mathcal{E}_N^{\text{rel}}(F) \quad \text{for every symmetric } F,$$

which would imply

$$(68) \quad \lambda_{N,\text{sym}}^{\text{LSI}} = 2N.$$

The radial family shows that the constant in (67), if true, is sharp but is not attained in the relative sector.

Under the Vandermonde transform, (67) becomes a ground-state logarithmic Sobolev inequality comparing entropy relative to the antisymmetric holomorphic ground state Δ_N with the anti-holomorphic Fock energy. The Poincaré proof does not tensorize to entropy: the estimate that at most half of the L^2 mass is holomorphic has no direct entropy analogue.

9.2. Degeneration certificate. A positive answer to degeneration requires symmetric probability densities q_N with

$$(69) \quad \frac{\mathcal{I}_{\nu_N^{\text{rel}}}(q_N)}{2 \text{Ent}_{\nu_N^{\text{rel}}}(q_N)} \longrightarrow 0.$$

Since perturbative and origin-radial constructions are ruled out, the natural adversarial branch is an entropy-capacity analysis of nonradial symmetric rare events. Probability estimates alone do not suffice: each candidate must include the cost of a Sobolev transition layer.

10. IMMEDIATE RESEARCH PROGRAM

The next work should proceed in two parallel subbranches.

- (1) **Ground-state branch.** Seek a sharp entropy inequality for the real symmetric quotient of the antisymmetric Fock sector. The first concrete target is a semigroup or entropy-production formula whose quadraticization reproduces the half-holomorphic-mass argument but which remains nonlinear.
- (2) **Capacity branch.** Compute Fisher/entropy quotients for explicit centered symmetric families involving angular–radial coupling: holes and overcrowding relative to the empirical center, anisotropic outliers, collision neighborhoods, and soft shape order statistics. Origin-radial and total-radius tilts are mandatory controls and give the nondegenerate scale $2N$.

Neither subbranch should inherit a preferred outcome. A proof of (67), an order- N lower bound with a nonsharp constant, an intermediate scale, and a genuine degeneration sequence are all logically open at this stage.

11. RELATION WITH EXISTING MECHANISMS

For one-dimensional beta-Hermite ensembles, convexity and exact factorization give sharp Poincaré and logarithmic Sobolev inequalities at the confinement scale, with the center coordinate furnishing the extremizers. The planar Ginibre interaction does not satisfy the same real convexity hypothesis. Recent transport work for two-dimensional Coulomb gases exploits log-subharmonicity and proves trace/volume-contraction and majorization statements, but a trace bound does not by itself transfer a logarithmic Sobolev inequality [3]. Thus neither route presently settles (66).

REFERENCES

- [1] D. Chafaï and J. Lehec, *On Poincaré and logarithmic Sobolev inequalities for a class of singular Gibbs measures*, in Geometric Aspects of Functional Analysis, Lecture Notes in Math. 2256, Springer, 2020, 219–246; arXiv:1805.00708.
- [2] E. Kostlan, *On the spectra of Gaussian matrices*, Linear Algebra Appl. **162–164** (1992), 385–388.
- [3] G. De Philippis and Y. Shenfeld, *Optimal transport maps, majorization, and log-subharmonic measures*, arXiv:2411.12109, version 4, 2026.